SUMMER SCHOOL : HIGGS BUNDLES ON P-ADIC CURVES AND REPRESENTATION THEORY

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In 1965, generalising and specifying a former result of Weil, Narasimhan and Seshadri [20] established a bijective correspondence between the set of equivalence classes of irreducible unitary representations of the fundamental group of a compact Riemann surface X of genus ≥ 2, and the set of isomorphism classes of degree 0 stable vector bundles on X. The correspondence was further extended to any complex projective smooth variety by Donaldson [9]. The analogue for general linear representations is due to Simpson, following works by Corlette and Donaldson ; in order to obtain a correspondence of the same type than the one by Narasimhan and Seshadri, we need to add an extra structure to the vector bundle. It's the notion of *Higgs bundle*, introduced first by Hitchin for algebraic curves. If X is a scheme, $\mathscr{E}$ an $\mathscr{O}_X$-module and $\mathscr{A}$ an $\mathscr{O}_X$-algebra, an *$\mathscr{A}$-Higgs module with coefficients in $\mathscr{E}$* is a pair $(\mathscr{M}, \theta)$ made of an $\mathscr{A}$-module $\mathscr{M}$ and an $\mathscr{A}$-linear morphism $\theta\colon \mathscr{M} \to \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{E}$ such that $\theta \wedge \theta = 0$. When X is a complex projective smooth variety, which is the case considered by Simpson, one exclusively considers $\mathscr{O}_X$-Higgs bundles with coefficients in $\Omega^1_{X/\mathbb{C}}$; we will simply call them Higgs bundles. But we need later to consider more general Higgs modules.

Simpson's main result [24, 25, 26, 27] establishes equivalences of categories between the category of (finite dimensional complex) linear representations of the fundamental group of a complex smooth projective variety X, the category of vector bundles with integrable connections on X and the category of semi-stable Higgs bundles on X with vanishing Chern classes (cf. [19]).

Simpson's results and subsequent developments have led since few years to the quest of a p-adic analogue. The most advanced approach at the present stage is due to Faltings [12]. It generalises former results by Tate, Sen and Fontaine, and relies on his theory of almost-étale extensions [11]. It is also a continuation of his works on p-adic Hodge theory, specially those establishing the existence of Hodge-Tate decompositions [10]. Once achieved, the p-adic Simpson correspondence should naturally give the best Hodge-Tate type statements in p-adic Hodge theory. At the present stage, Faltings construction appears to be satisfactory only for curves and, even in this case, many fundamental questions remain open.

Meanwhile, Deninger and Werner [7, 8, 6] developed a theory of parallel transport for vector bundles over p-adic curves, providing an analogue of the classical theory of Narasimhan and Seshadri.

The summer school at Mainz is an introduction to these p-adic theories, with a particular emphasis on Faltings' approach. There will be also few lectures on related topics in p-adic Hodge theory and non-abelian Hodge theory in characteristic p.

Faltings' approach

Let K be a complete discrete valuation field of characteristic 0, with perfect residue field k of characteristic $p > 0$, $\overline{K}$ an algebraic closure of K. We denote by $\mathscr{O}_K$ the valuation ring of K, by $\mathscr{O}_{\overline{K}}$ the integral closure of $\mathscr{O}_K$ in $\overline{K}$, by $\mathscr{O}_C$ the p-adic separated completion of $\mathscr{O}_{\overline{K}}$ and by C the field of fractions of $\mathscr{O}_C$. Let X be a proper smooth geometrically connected curve over K, $\overline{x}$ a geometric point of $X_{\overline{K}}$. Faltings constructs a fully faithful functor from the category of p-adic representations of $\pi_1(X_{\overline{K}}, \overline{x})$ to the category of $(\mathscr{O}_{X_C})$-Higgs bundles with coefficients in $\Omega^1_{X/K}(-1)$. This functor is in fact defined on a strictly larger category than that of p-adic representations of $\pi_1(X_{\overline{K}}, \overline{x})$, namely the category of *generalised representations* of $\pi_1(X_{\overline{K}}, \overline{x})$, and it induces then an equivalence of categories between this new category and that of $(\mathscr{O}_{X_C})$-Higgs bundles with coefficients in $\Omega^1_{X/K}(-1)$. Faltings shows that Higgs modules associated to "true" p-adic representations of $\pi_1(X_{\overline{K}}, \overline{x})$ are semi-stables of slope zero. It is expected that all slope zero semi-stable Higgs modules can be obtained in this way. This statement would correspond to the most difficult part of Simpson's result in the complex case.

As usually in his work in p-adic Hodge theory, Faltings proceeds by *glueing* local constructions on certain affine open subsets of an integral model of X on which he has a certain control and that he qualifies by *small*. The local aspect of the theory is in fact developed for schemes of any dimension.

Small representations. The typical example of a small affine scheme considered by Faltings is an affine scheme $\mathrm{Spec}(R)$, étale over the torus $\mathbb{G}^d_{m,\mathscr{O}_K} = \mathrm{Spec}(\mathscr{O}_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}])$ and having integral geometric fibres over $\mathrm{Spec}(\mathscr{O}_K)$. The fundamental group Δ of $\mathrm{Spec}(R_{\overline{K}})$ (relatively to a geometric generic point) can be explicitly described as follows. Let F be the fraction field of R, F^a an algebraic closure of F containing $\overline{K}$, $\overline{F}$ the union of all finite extensions L of F contained in F^a such that the integral closure of R in L is étale over $\mathrm{Spec}(R_K)$, $\overline{R}$ the integral closure of R in $\overline{F}$ and $R_1 = R \otimes_{\mathscr{O}_K} \mathscr{O}_{\overline{K}}$. Then Δ is the Galois group of $\overline{F}$ over $F \otimes_K \overline{K}$. In this setting, *generalised representations* of Δ are continuous semi-linear representations of Δ on projective $\widehat{\overline{R}}$-modules of finite type (where for any $\mathbb{Z}_p$-module M, we denote $\widehat{M}$ its p-adic separated completion).

Faltings constructs generalised representations from certain Higgs modules. More precisely, let (M, θ) be a $\widehat{R_1}$-Higgs module with coefficients in $\Omega^1_{R/\mathscr{O}_K}(-1)$ such that the underlying $\widehat{R_1}$-module is free of finite type and that θ is divisible by $p^{2\alpha}$ for a rational number $\alpha > \frac{1}{p-1}$; such a Higgs module is called *small*. Faltings associates to (M, θ) a p-adically continuous semi-linear representation φ of Δ with values in $M \otimes_{\widehat{R_1}} \widehat{\overline{R}}$. The generalised representation thus obtained is *small* in the sense that it admits a $\widehat{\overline{R}}$-basis of elements that are invariant modulo $p^{2\alpha} M \otimes_{\widehat{R_1}} \widehat{\overline{R}}$. As a consequence of his purity theorem for almost étale extensions, Faltings proves that this construction yields an equivalence of categories between the category of small generalised representations of Δ and that of small $\widehat{R_1}$-Higgs module with coefficients in $\Omega^1_{R/\mathscr{O}_K}(-1)$. It is the corner stone of the p-adic Simpson correspondence. The correspondence is moreover compatible with the natural cohomological functors : up to torsion, the continuous cohomology of the group Δ with coefficients in $M \otimes_{\widehat{R_1}} \widehat{\overline{R}}$ is isomorphic to the cohomology of the Koszul complex of (M, θ) (also called the *Dolbeault* complex of (M, θ)).

Restricting to curves, Faltings develops also the notions of *small* Higgs modules and *small* generalised representations with rational coefficients. He obtains again an equivalence of categories. These results are more generally true for small affine schemes endowed with appropriate log structures.

The most crucial point in this construction, aside from the purity theorem, is the control of its degree of canonicity. In fact, Faltings gives two constructions, the most useful and better suited for globalisation depends on the data of a deformation of the small affine scheme over a certain infinitesimal thickening $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$ of $\mathscr{O}_C$ formerly defined by Fontaine. Inspired by this construction, but also by the work of Ogus and Vologodsky on an analogue of Simpson's correspondence in characteristic p, Abbes and Gros [2] develop another approach to p-adic Simpson correspondence that generalises Faltings' approach and that clarifies the relation with Hyodo's work on Hodge-Tate local systems [17].

Glueing. The correspondence between small generalised representations and small Higgs modules described above can be extended for any scheme X over $\mathscr{O}_K$ endowed with an appropriate logarithmic structure, if we are given a logarithmic deformation of $X \otimes_{\mathscr{O}_K} \mathscr{O}_C$ over $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$. For this purpose, Faltings uses a topos that he introduced for the proof of the comparison theorems in p-adic Hodge theory [11]. It turns out that one needs a slightly different version, namely the *co-vanishing* topos, which is a variant of oriented products of topos defined by Deligne [3, 18].

Descent. It remains to remove the assumption of smallness. Faltings describes only the case of curves. To simplify the exposition, let's consider a proper semi-stable curve X over $\mathscr{O}_K$ and fix a logarithmic deformation of $X \otimes_{\mathscr{O}_K} \mathscr{O}_C$ over $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$. Starting from of a general $(\mathscr{O}_{X_C})$-Higgs module (E, θ) with coefficients in $\Omega^1_{X/\mathscr{O}_K}(-1)$, Faltings shows that, after a finite extension of K, one can always find a proper morphism $f: Y \to X$ such that Y is a semi-stable curve over $\mathscr{O}_K$, f_K is a Galois covering and the inverse image $f^*(E, \theta)$ is small. To this Higgs module is henceforth associated a small representation $\mathscr{M}$ on Y, once fixed a deformation of $Y \otimes_{\mathscr{O}_K} \mathscr{O}_C$ over $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$ (which is always possible). It should be then sufficient to descend $\mathscr{M}$. But *a priori*, there is no natural descent data on $\mathscr{M}$. To produce such a data, Faltings twists $f^*(E, \theta)$ by tensor product with an invertible module $\mathscr{L}$. This new module remains small, and the small corresponding generalised representation is naturally endowed with a descent data. So, it descends to X producing the sought-after functor. The need of twisting the inverse image of Higgs modules is related to the fact that f does not necessarily lift to deformations over $\mathscr{A}_2(\mathscr{O}_{\overline{K}})$. The obstruction for the existence of such a lift intervenes in the definition of $\mathscr{L}$. One also needs in the construction to choose an exponential map on the multiplicative group. Faltings finally shows that one obtains an equivalence of categories between the category of $(\mathscr{O}_{X_C})$-Higgs modules (E, θ) with coefficients in $\Omega^1_{X/\mathscr{O}_K}(-1)$ and that of generalised representations on X with rational coefficients.

Deninger and Werner's approach

The theory of Deninger and Werner equips pss-vector bundles E of slope zero on smooth projective curves X over p-adic fields with a functorial parallel transport along "étale paths". Restricted to closed paths, one obtains representations of the fundamental group on the fibres of the vector bundle. Here pss stands for "potentially strongly semi-stable reduction", a somewhat technical notion. The pss-bundles of slope zero on curves form neutral Tannakian categories which are stable under pullback by finite morphisms. Every pss-bundle is semi-stable and the main open question is whether the converse holds. If one views a pss-bundle as being equipped with the zero Higgs field, then the associated representation of the fundamental group on a fibre corresponds to the inverse of Faltings' construction. However the parallel transport between different fibres provides a stronger structure. Other topics of interest :

- A comparison of the reduction of the p-adic representation attached to (a model) of E with the representation of the Nori fundamental group (of the special fibre) corresponding to the reduction of E.
- For pss-bundles of non-zero slope, the construction of a corresponding p-adic representation of a certain central extension of the fundamental group.

Reading suggestions for young participants

Beyond the classical theories of étale fundamental groups [16] and local fields [23], the prerequisites for young participants who would like to follow the summer school are on the one hand, the arithmetic theory of p-adic Galois representations as in [15], and on the other hand, the semi-stable reduction theorem for curves [1]. Further readings on p-adic Hodge theory may be useful [10, 11, 14]. For Scholze's lectures [22], one can consult the video of his lectures at IHÉS on *Perfectoid Spaces and the Weight-Monodromy Conjecture* (http ://www.ihes.fr/∼abbes/CAGA/scholze.html).

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