# Parametric Renormalization in Scalar and Gauge Fields

Dirk Kreimer, Humboldt University, Berlin

Supported by an Alexander von Humboldt Chair by the Alexander von Humboldt Foundation and the BMBF

based on joint work with

**F.Brown (1112.1180); M.Sars, W.van Suijlekom (1208.6477); K.Yeats (1207.5460)**

**Feynman graphs and motives, Bingen, March 18-22, 2013**

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- The coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \overbrace{\sum_{\gamma=\cup_i \gamma_i, \omega_4(\gamma_i) \geq 0} \gamma \otimes \Gamma/\gamma}^{\Delta'(\Gamma)} \qquad (1)$$

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- The antipode

$$S(\Gamma) = -\Gamma - \sum S(\gamma)\Gamma/\gamma = -m(S \otimes \mathrm{P})\Delta \tag{2}$$

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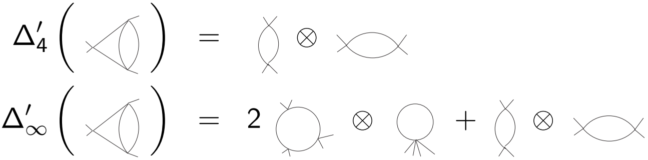
- The renormalized Feynman rules

$$\Phi_R = m(S_R^\Phi \otimes \Phi)\Delta \quad (5)$$

# BCFW and the core Hopf algebra

- a sequence of quotient Hopf algebras by looking at short distance singularities in $2k$ dimensions

$$H_0 \subset H_2 \subset H_4 \subset \cdots \subset H_{2k} \subset \cdots H_\infty \tag{6}$$



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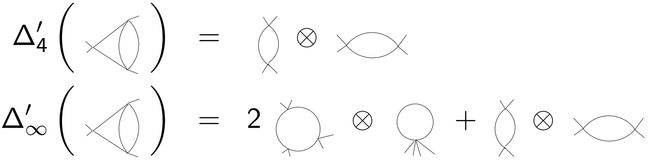

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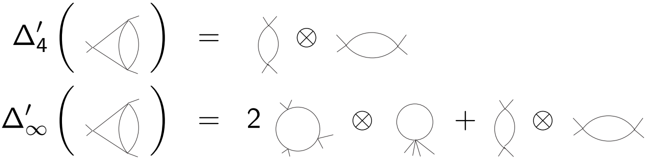


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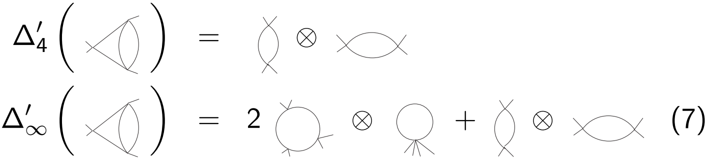


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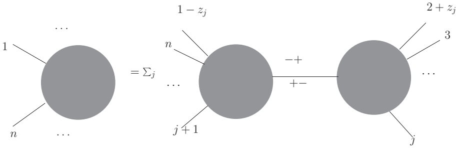

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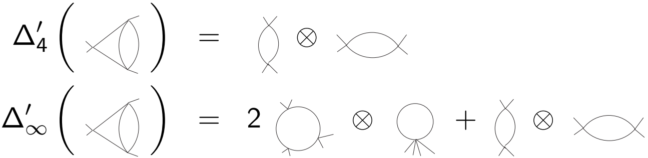


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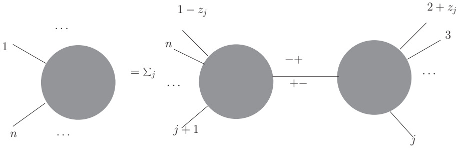


- KLT relations or kinematic STU ↔ co-ideal respected

# Kinematics and Cohomology

- Exact co-cycles

$$[B_+^{r,j}] = B_+^{r;j} + b\phi^{r;j} \tag{8}$$

with $\phi^{r;j} : H \to \mathbb{C}$

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- Variation of momenta

$$G^R(\{g\}, \ln s, \{\Theta\}) = 1 \pm \Phi^R_{\ln s,\{\Theta\}}(X^r(\{g\})) \tag{9}$$

with $X^r = 1 \pm \sum_j g^j B_+^{r;j}(X^r Q^j(g))$, $bB_+^{r;j} = 0$. Note:
$\beta(g) = 0 \Leftrightarrow Q(g) = \mathrm{constant}$.
Then, for kinematic renormalization schemes:
$\{\Theta\} \to \{\Theta'\} \Leftrightarrow B_+^{r;j} \to B_+^{r;j} + b\phi^{r;j}$.
$\Phi^R_{L_1+L_2,\{\Theta\}} = \Phi^R_{L_1,\{\Theta\}} \star \Phi^R_{L_2,\{\Theta\}}$.
$\Phi^R(\ln s, \{\Theta\}, \{\Theta_0\}) = \Phi^{-1}_{\mathrm{fin}}(\{\Theta_0\} \star \Phi^R_{1-\mathrm{scale}}(\ln s) \star \Phi_{\mathrm{fin}}(\{\Theta\})$.

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# Graph Polynomials

- $$0 \to H^1(\Gamma) \to \mathbb{Q}^{E_\Gamma} \to \mathbb{Q}^{V_\Gamma,0} \to 0. \qquad (10)$$

  $\{h_i\}$ basis of homology (loops!)

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- $$N := \begin{pmatrix} N_0 := \left(\sum_{e \in h_i \cap h_j} A_e\right)_{ij} \mathbb{I} & \sum_{e \in h_j} \mu_e A_e \\ \sum_{e \in h_j} \bar{\mu}_e A_e & \sum_{e \in \Gamma^{[1]}} \bar{\mu}_e \mu_e A_e \end{pmatrix}$$

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- $|N_0| = \psi(\Gamma)$,
  $|N| = \phi(\Gamma) := -\sum_{T_1\cup T_2} \sum_{e\notin T_1\cup T_2} (\sigma(e) q_e)^2 \prod_{e\notin T_1\cup T_2} A_e$.

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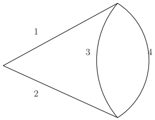
# Example

$\Gamma =$




$$N_\Gamma = \begin{pmatrix} A_1 + A_2 + A_3 & A_1 + A_2 & A_1\mu_1 + A_2\mu_2 + A_3\mu_3 \\ A_1 + A_2 & A_1 + A_2 + A_4 & A_1\mu_1 + A_2\mu_2 + A_4\mu_4 \\ A_1\bar{\mu}_1 + A_2\bar{\mu}_2 + A_3\bar{\mu}_3 & A_1\bar{\mu}_1 + A_2\bar{\mu}_2 + A_4\bar{\mu}_4 & \sum_{i:=1}^4 A_i\bar{\mu}_i\mu_i \end{pmatrix}$$

$$\psi_\Gamma = (A_1 + A_2)(A_3 + A_4) + A_3A_4 = \sum_{\mathrm{sp.Tr.}T} \prod_{e\notin T} A_e$$

$$\phi_\Gamma = (A_3 + A_4)A_1A_2p_a^2 + A_2A_3A_4p_b^2 + A_1A_3A_4p_c^2 = \sum_{\mathrm{sp.2-Tr.}T_1\cup T_2} Q(T_1)\cdot Q(T_2)\prod_{e\notin T_1\cup T_2} A_e.$$

# The Feynman rules in projective space

First, $\phi_\Gamma \to \phi_\Gamma + \psi_\Gamma(\sum_e m_e^2 A_e)$.

$$\Phi_\Gamma^R(S, S_0, \{\Theta, \Theta_0\}) = \int_{\mathbb{P}^{E-1}(\mathbb{R}_+)} \overbrace{\sum_f}^{\text{forestsum}} (-1)^{|f|} \frac{\ln \frac{\frac{S}{S_0}\phi_{\Gamma/f}\psi_f + \phi_f^0 \psi_{\Gamma/f}}{\phi_{\Gamma/f}^0 \psi_f + \phi_f^0 \psi_{\Gamma/f}}}{\psi_{\Gamma/f}^2 \psi_f^2} \underbrace{\Omega_\Gamma}_{(E-1)-\text{form}}$$

Note: for 1-scale graphs, $\phi_\Gamma = \psi_\Gamma^\bullet$.

# The polylog as a Hodge structure

Iterated integrals: obvious Hopf algebra structure

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2\pi i & 0 \\ -Li_2(z) & 2\pi i \ln(z) & (2\pi i)^2 \end{pmatrix} = (C_1, C_2, C_3) \qquad (11)$$

$$\mathrm{Var}(\Im Li_2(z) - \ln|z|\,\Im Li_1(z)) = 0 \qquad (12)$$

Hodge sructure from Hopf algebra structure: branch cut ambiguities columnwise

Griffith transversality $\Leftrightarrow$ differential equation

# Limiting mixed Hodge structures

- Hopf algebra from flags

$$f := \gamma_1 \subset \gamma_2 \subset \ldots \subset \Gamma,\ \Delta'(\gamma_{i+1}/\gamma_i) = 0 \qquad (13)$$

The set of all such flags $F_\Gamma \ni f$ determines Hopf algebra structure, $|F_\Gamma|$ is the length of the flag.

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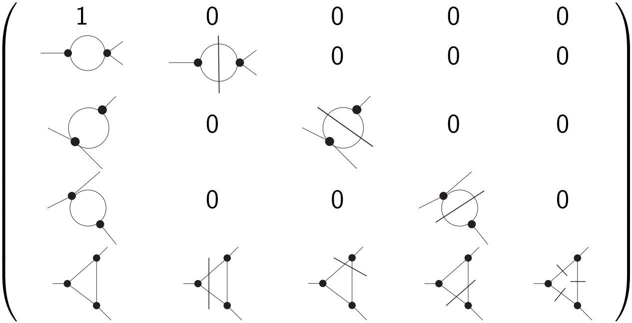
- It also determines a column vector $v = v(F_\Gamma)$ and a nilpotent matrix $(N) = (N(|F_\Gamma|))$, $(N)^{k+1} = 0$, $k = corad(\Gamma)$ such that

$$\lim_{t\to 0} (e^{-\ln t (N)}) \Phi_R(v(F_\Gamma)) = (c_1^\Gamma(\Theta)\ln s, c_2^\Gamma(\Theta), c_k^\Gamma(\Theta)\ln^k s)^T \tag{14}$$

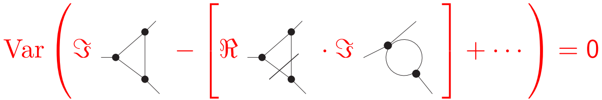
where $k$ is determined from the co-radical filtration and $t$ is a regulator say for the lower boundary in the parametric representation.

# The Feynman graph as a Hodge structure

Hopf algebra structure as above




$= (C_1, C_2, C_3, C_4, C_5)$




Hodge structure: cut-reconstructability: from Hopf algebra structure:
branch cut ambiguities columnwise
Griffith transversality ⇔ differential equation?

$$\zeta(s_1, \cdots, s_k) = \sum_{n_i < n_{i+1}} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

- counting over $\mathbb{Q}$

$$1 - \frac{x^3 y}{1 - x^2} + \frac{x^{12} y^2 (1 - y^2)}{(1 - x^4)(1 - x^6)} = \prod_{n \geq 3} \prod_{k \geq 1} (1 - x^n y^k)^{D_{n,k}} \quad (15)$$

$\rightarrow$ first irreducible MZV from planar graphs at 7 loops in scalar field theory (integrability???)

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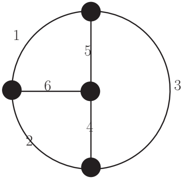
- When is a graph redicible to MZVs? Francis Brown: when it has vertex width three.
- Caution! Non-MZVs at eight loops from non-planar graphs, at nine loops from planar graphs ('A $K3$ in $\phi^4$', Brown and Schnetz). Proof from counting points $[X_\Gamma]$ on graph hypersurfaces $X_\Gamma$ over $\mathbb{F}_q$, defined by vanishing of the first Symanzik polynomial. If the graph gives a MZV, $[X_\Gamma]$ better is polynomial in the prime power $q = p^n$. Alas, it is not in general, with counterexamples relating graphs to elliptic curves with complex multiplication, and point-counting function a modular form.

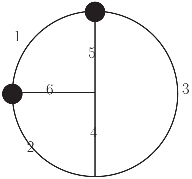
# Decomposing scales and angles

Consider

$\Gamma =$  , (16)



and

$\Gamma^2 =$  . (17)



We let $S = p_1^2 + p_2^2 + p_3^2 + 2p_1 \cdot p_2 + 2p_2 \cdot p_3 + 2p_3 \cdot p_1$ (which defines the variable angles $\Theta^{ij} = p_i \cdot p_j / S, \Theta^e = m_e^2/S$) and subtract symmetrically say at $S_0, \Theta_0^{ij} = \frac{1}{3}(4\delta_{ij} - 1)$ and $\Theta_0^e = m_e^2/S_0$, which specifies the fixed angles $\Theta_0$.

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$$\Phi^R_\Gamma = \frac{\ln \frac{\frac{S}{S_0}\phi_\Gamma(\Theta)}{\phi_\Gamma(\Theta_0)}}{\psi^2_\Gamma}\Omega_\Gamma. \tag{18}$$

To find the desired decomposition, we use

$$\Delta^2(\Gamma) = \Gamma \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \Gamma. \tag{19}$$

We then have

$$\Phi^R_\Gamma = \Phi^{-1}_{\mathrm{fin}}(\Theta_0)(\Gamma) + \Phi^R_{1-\mathrm{s}}(S/S_0)(\Gamma) + \Phi_{\mathrm{fin}}(\Theta)(\Gamma). \tag{20}$$

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We have

$$\Phi_{\mathrm{fin}}^{-1}(\Theta_0)(\Gamma) = -\frac{\ln \frac{\phi_\Gamma(\Theta_0)}{\psi_{\Gamma^2\bullet}}}{\psi_\Gamma^2}\Omega_\Gamma, \tag{21}$$

$$\Phi_{1-\mathrm{s}}^R(S/S_0)(\Gamma) = \frac{\ln \frac{S}{S_0}}{\psi_{\Gamma^2}^2}\Omega_\Gamma, \tag{22}$$

which integrates to the renormalized value $\Phi_{1-\mathrm{s}}^R(S/S_0)(\Gamma) = 6\zeta(3)\ln\frac{S}{S_0}$. Finally,

$$\Phi_{\mathrm{fin}}(\Theta)(\Gamma) = \frac{\ln \frac{\phi_\Gamma(\Theta)}{\psi_{\Gamma^2\bullet}}}{\psi_\Gamma^2}\Omega_\Gamma. \tag{23}$$

These integrands indeed all converge, which is synonymous for us to say that they can be integrated against $\mathbb{P}^{E-1}(\mathbb{R}_+)$.

# From scalar 3-regular graphs to gauge theory amplitudes

Needed:

- The corolla polynomial and differentials

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# From scalar 3-regular graphs to gauge theory amplitudes

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- The corolla polynomial and differentials
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- Previous set-up: Kirchhoff polynomials for 3-regular scalar graphs
- Then: The renormalized Feynman integrand of gauge theory from the sum of all 3-regular connected graphs.

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# The corolla polynomial

We now consider a polynomial based on half-edges. We need the following definitions

- For a vertex $v \in V$ let $n(v)$ be the set of edges incident to $v$ (internal or external).

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- For $i \geq 0$ let

$$C^i_{3g} = \sum_{\substack{C_1, C_2, \ldots C_i \in \mathcal{C} \\ C_j \text{pairwise disjoint}}} \left( \left( \prod_{j=1}^{i} \prod_{v \in C_j} a_{v, v_C} \right) \prod_{v \notin C_1 \cup C_2 \cup \cdots \cup C_i} D_v \right)$$

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# The corolla polynomial

We now consider a polynomial based on half-edges. We need the following definitions

- For a vertex $v \in V$ let $n(v)$ be the set of edges incident to $v$ (internal or external).
- For a vertex $v \in V$ let $D_v = \sum_{j \in n(v)} a_{v,j}$.
- Let $\mathcal{C}$ be the set of all cycles of $G$ (cycles, not circuits). This is a finite set.
- For $C$ a cycle and $v$ a vertex in $V$, since $G$ is 3-regular, there is a unique edge of $G$ incident to $v$ and not in $C$, let $v_C$ be this edge.
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- Let

$$C_{3g} = \sum_{j \geq 0} (-1)^j C^j_{3g}$$

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This is a polynomial -the corolla polynomial- because $C^i_{3g} = 0$ for $i > |\mathcal{C}|$.
We write $C^\Gamma_{3g}$ for the corolla polynomial of a 3-regular connected graph $\Gamma$.

## Theorem

*Let $\mathcal{T}$ be the set of sets $T$ of half edges of $G$ with the property that*

- *every vertex of $G$ is incident to exactly one half edge of $T$*
- *$G \setminus T$ has no cycles*

*Then*

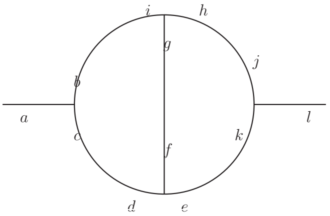
$$C_{3g} = \sum_{T \in \mathcal{T}} \prod_{h \in T} a_h$$

More properties joint with Karen Yeats.

# Example

Look at

$\Gamma =$




$$
\begin{aligned}
C_{3g}(\Gamma) &= (a+b+c)(d+e+f)(i+g+h)(j+k+l) \\
&- (aeh)(j+k+l) \\
&- (lid)(a+b+c) \\
&- (algf) \qquad (24)
\end{aligned}
$$

# The corolla differentials

Our main use of the corolla polynomial is to construct differential operators with it.
These operators differentiate wrt momenta $\xi(e)$ assigned to edges $e$ of a graph, and act on the second Kirchhoff polynomial written for generic edge momenta $\xi(e)$.
Only at the end of the computation will we determine the $\xi(e)$ so that the agree with external momenta.

For a half edge $h \equiv (w, f) \in H^\Gamma$, we let $e(h) = f$ and $v(h) = w$.
$h_+$ and $h_-$ are the successor and the precursor of $h$ in the oriented corolla at $v(h)$.
We assign to a graph $\Gamma$:

- i) to each (possibly external) edge $e$, a variable $A(e)$ and a 4-vector $\xi(e)$;

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- ii) to each half edge $h$, a Lorentz index $\mu(h)$;
- iii) to each corolla, a representation of the Lie algebra of the gauge group.

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# Gluon-ghost and fermion differentials

Furthermore, we assign to each $h \in H^\Gamma$ two types of differential operators: either the differential operator $D_g(h)$,

$$D_g(h) := \left( \frac{1}{A_{e(h_+)}} \frac{\partial}{\partial \xi(e(h_+))_{\mu(h)}} - \frac{1}{A_{e(h_-)}} \frac{\partial}{\partial \xi(e(h_-))_{\mu(h)}} \right) g_{\mu(h_+)\mu(h_-)}.$$

Or the differential operator

$$D_f(h) := \left( \frac{1}{A_{e(h_+)}} \frac{\partial}{\partial \xi(e(h_+))_{\mu(h_+)}} \gamma_{\mu(h_+)} - \frac{1}{A_{e(h_-)}} \frac{\partial}{\partial \xi(e(h_-))_{\mu(h_-)}} \gamma_{\mu(h_-)} \right) \gamma_{\mu(h)}.$$

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# Graph differentials

The corolla polynomial is an alternating sum $(-1)^j C^j_{3g}$.
It depends on half-edge variables $a_h$.
For a collection of cycles $C_1, \cdots, C_j$ contributing to $C^j_{3g}$, consider partitions of this set into two subsets $I_k, I_l$ containing $k + l = j$ cycles.
Replace $a_{v,v_C} \to b_{v,v_C}$ for each $C \in I_l$. This defines $C^{I_k,I_l}_{3g}(\Gamma)(a_h, b_h)$.
Sum over all possible partitions $I_k, I_l$ of the cycles for each $j$. This gives a further corolla polynomial for which we write in slight abuse of notation $C_{3g}(\Gamma)(a_h, b_h)$.
Assign a differential operator as follows:

$$C_{3g}(\Gamma)(a_h, b_h) \to D^{\mathrm{gauge}}(\Gamma) = \mathrm{colour}^{I_k,I_l}(\Gamma) \sum_{j\geq 0} \sum_{|I_k|+|I_l|=j} \tilde{C}^{I_k,I_l}_{3g}(\Gamma)(D_g(h), D_f(h)). \tag{25}$$

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# cont'd

Note that the restriction to $I_l = \emptyset$ gives the corresponding operator for Yang-Mills theory.

$C_{3g}^{I_k,I_l}(\Gamma)(D_g(h), D_f(h))$ is a homogeneous differential operator of degree $|V^\Gamma|$ which is at most quadratic in each derivative $\partial_{\xi(e)}$, $e \in E^\Gamma$.

For non-empty $I_k \cup I_l$, let $\tilde{C}_{3g}^{I_k,I_l}$ be the part of $C_{3g}^{I_k,I_l}$ which is linear in each variable $1/A_e$. Set $\tilde{C}_{3g}^{\emptyset,\emptyset} = C_{3g}^{\emptyset,\emptyset}$ else.

For $k$ open ghost lines and $j$ open fermion lines there is a similar definition available.

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# 3-regular graphs

This is joint with Matthias Sars and Walter van Suijlekom.

- We start with connected 3-regular graphs. To a graph with $k$ external edges, we assign a powercounting weight $\omega_\Gamma = 4 - k$.
  $\Gamma$ is convergent for $k > 4$.
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- We also consider pairs of a graph together with a collection of (its) disjoint cycles, and filter such pairs by the number of cycles.
- Similar if we allow for 4-valent vertices, we also filter by the number of such vertices.

# Cycle Homology

The corolla polynomial is a sum of half-edge variables such that the variables not in any contributing monomial do not correspond to a cycle in the graph.
This allows to consider a 3-regular graph together with its set of (disjoint) cycles.

$$\Gamma \to (\Gamma, C_{i_1} \cdots C_{i_k}).$$

Disjointness makes this interesting.
The corolla polynomial eliminates all pairs but $(\Gamma, \emptyset)$ (ghost loops eliminate gaugeons).
This suggests $t := \sum_i \partial_{C_i}$ where we can treat the $C_i$ as formal Grassmann variables. Together with graph homology this leads to a double complex which ensures gauge invariant amplitudes.

# Graph Homology

For an edge $e$ in a graph $\Gamma$, let $\Gamma_e$ be the graph where $e$ shrinks to zero length.
Its orientation is obtained as follows:
we permute vertex labels collecting signs until the edge $e$ connects vertex 1, $s(e) = 1$, to vertex 2, $t(e) = 2$.
Let $\sigma$ be the sign of the necessary permutations.
Then we shrink edge $e$ and the so-obtained vertex is labelled 1.
We inherit all remaining edge orientations and the ordering of vertices remains unchanged, with vertices $3, 4, \ldots, |V^\Gamma|$ relabelled to $2, 3, \ldots, |V^\Gamma| - 1$.
This defines an orientation of $\Gamma_e$.
If $\sigma$ is negative, we change the orientation by en edge swap.

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# cont'd

For an oriented graph $\Gamma$, let

$$s\Gamma = \sum_{e \in E_I} \Gamma_e, \qquad (26)$$

be a sum of graphs obtained by shrinking edge $e$ and assigning the orientation as above. Graph homology comes from the classical result

**Theorem**

*(graph homology)* $s \circ s = 0$.

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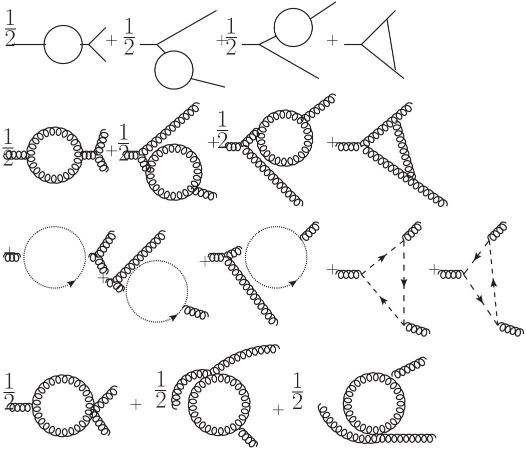
# Graph homology and the residue

Note that we integrate against the simplex $\sigma_\Gamma$ with boundary $\prod_e A_e = 0$.
We have co-dimension $k$-hypersurfaces given by $A_{i_1} = \cdots = A_{i_k} = 0$.
The Feynman integrand we have constructed above comes from a regular parts, and from residues along these hypersurfaces.
It can be described by the following commutative diagram.

$$\begin{array}{ccc} \Gamma & \xrightarrow{s=\sum_e s_e} & s\Gamma \\ \downarrow \Phi & & \downarrow \Phi \\ \Phi(\Gamma) & \xrightarrow{\sum_e \mathrm{Res}_e} & \Phi(s\Gamma) \end{array}$$

# Example



# The renormalized result

### Theorem

*The unrenormalized Feynman integrand at n loops for the sum of all Feynman graphs contributing to the connected k-loop amplitude is* $\Phi(\Gamma^k) =$

$$\sum_{|\Gamma|=n,|E_E(\Gamma)|=k} e^{-\sum_e \oint_{c_e}} \left(\prod_{e\in E^\Gamma} g_{\mu(v_1(e))\mu(v_2(e))} D_{\mathrm{hom}}^{\mathrm{gauge}}(\Gamma)\right) \frac{e^{-\frac{\phi_\Gamma}{\psi_\Gamma}}}{\psi_\Gamma^2} \prod_{e\in E^\Gamma} dA_e.$$

The renormalized result is obtained as

$$D_{\mathrm{hom}}^{\mathrm{gauge}} \sum_{f\in\mathcal{F}} (-1)^{|f|} \frac{e^{-\frac{\phi_{\Gamma/f}}{\psi_{\Gamma/f}}}}{\psi_{\Gamma/f}^2} \frac{e^{-\frac{\phi_f}{\psi_f}}}{\psi_f^2}$$

with the graph differential in front of the forest sum.

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# Remarks

- Ghosts and fermions automated in corolla polynomial, transversal and longitudinal contributions not independent as derivatives $\partial_{\xi(e)}$ are at most second order.

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- Spin 1/2, spin 1 from scalar 3-regular graphs and restricted graph homology. What about spin 2, gravity?

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