Elliptic integrals in quantum field theory

David Broadhurst, [a,b] **Bingen, 18–22 March 2013**

Massless diagrams in quantum field theory often, yet not invariably, evaluate to polylogarithms. With massive propagators, elliptic integrals appear as early as two loops. They are not to be feared, but rather welcomed, since elliptic integrals have amazingly fast numerical evaluations. In the first talk, I shall consider some fairly well-known appearances of elliptic integrals in QFT. In the second, I shall move on to diagrams that evaluate to products of elliptic integrals, which pave the way for more general evaluations, as the L-functions of modular forms at integers inside Dirichlet's critical strip, which likewise have very fast evaluations.

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1 Two-loop two-point function

The two-loop diagram, with scalar propagators, leads to an integral which we normalize as follows:

$$I(q^2) \equiv -\frac{q^2}{\pi^4} \int d^4l \int d^4k \; P_1(l) P_2(l-q) P_3(l-k) P_4(k) P_5(k-q) \tag{1}$$

where $P_i(l) \equiv (l^2 - m_i^2 + \mathrm{i}\epsilon)^{-1}$. The function (1) is analytic in the q^2-plane cut along the positive real axis, with the lowest branchpoint at

$$s_0 = \mathrm{Min}(\; [m_1+m_2]^2 \;, \; [m_4+m_5]^2 \;, \; [m_2+m_3+m_4]^2 \;, \; [m_1+m_3+m_5]^2 \;)$$

It vanishes at the origin (unless all the masses vanish) and is bounded at infinity, since $I(-\infty) = 6\zeta(3)$. The discontinuity across the cut is given by

$$\begin{aligned} \sigma(w^2) &\equiv \frac{1}{\pi} \,\mathrm{Im}\, I(w^2 + \mathrm{i}\epsilon) \\ &= \Big\{ \Theta(w - m_1 - m_2) \sigma_a(w^2) + (1 \leftrightarrow 4, 2 \leftrightarrow 5) \Big\} \\ &\quad + \Big\{ \Theta(w - m_2 - m_3 - m_4) \sigma_b(w^2) + (1 \leftrightarrow 2, 4 \leftrightarrow 5) \Big\} \end{aligned}$$

where $\sigma_{a,b}$ correspond to two-particle and 3-particle cuts.

The contribution σ_a, with a two-particle cut, involves a form factor that can in turn be evaluated dispersively by a two-particle cut (provided there is no anomalous threshold). This yields an inverse hyperbolic tangent in the discontinuity of the form factor, after integration over the momentum transfer in the scattering process $1+2 \to 4+5$, resulting in

$$\sigma_a(w^2) = -\int_{m_4+m_5}^{\infty} dx \frac{4x}{x^2 - w^2} \frac{\Delta(w^2, m_1^2, m_2^2)}{\Delta(x^2, m_1^2, m_2^2)} T(x^2, m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \tag{2}$$

where the principal value is to be taken and

$$\begin{aligned}\Delta(a,b,c) &\equiv \left\{a^2+b^2+c^2-2(ab+bc+ca)\right\}^{\frac{1}{2}}\\ T(s,a,b,c,d,e) &\equiv \tanh^{-1}\left(\frac{\Delta(s,a,b)\Delta(s,d,e)}{s^2-s(a+b-2c+d+e)+(a-b)(d-e)}\right)\end{aligned}$$

To evaluate the contribution σ_b, with a 3-paerticle cut, we must integrate over the Dalitz plot for the process $w \to m_2 + m_3 + m_4$. The integration over the invariant mass in the 2+3 channel also gives a $\tanh^{-1}$ function, which must then be integrated over the invariant mass in the 3+4 channel, giving

$$\sigma_b(w^2) = \int_{m_3+m_4}^{w-m_2} dx \frac{4x}{x^2-m_1^2} T(x^2, w^2, m_2^2, m_5^2, m_4^2, m_3^2) \tag{3}$$

The $\tanh^{-1}$ functions can now be removed from each of the integrands of eqs (2,3) by taking the derivative of σ, which is sufficient to evaluate the diagram (1) from the dispersion relation

$$I(q^2) = -\int_{s_0}^{\infty} ds\, \sigma'(s) \left\{\log\left(1-\frac{q^2}{s}\right) - \log\left(1-\frac{q^2}{s_0}\right)\right\} \tag{4}$$

where the second logarithm may be dropped if $\sigma(s_0) = 0$ (as is the case when $s_0 = 0$).

The derivative of σ_a of eq (2) may be evaluated using

$$\frac{\partial}{\partial w}\left(\frac{4x}{x^2-w^2}\frac{\Delta(w^2,m_1^2,m_2^2)}{\Delta(x^2,m_1^2,m_2^2)}\right) = \frac{\partial}{\partial x}\left(\frac{4w}{w^2-x^2}\frac{\Delta(x^2,m_1^2,m_2^2)}{\Delta(w^2,m_1^2,m_2^2)}\right)$$

2 Three-loop massive bubble diagrams

There are 10 distinct colourings of the tetrahedron by mass, illustrated in Fig. 2. The massive lines in V_{2A} and V_{2N} are adjacent and non-adjacent, respectively; in the dual cases, V_{4A} and V_{4N}, it is the massless lines that are adjacent and non-adjacent; in cases V_{3T}, V_{3S} and V_{3L}, the massive lines form a triangle, star and line, and hence the massless lines form a star, triangle and line.

Fig 1: Symmetries of the tetrahedron

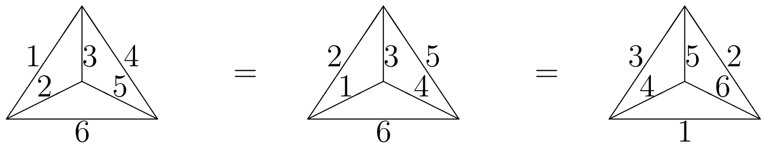
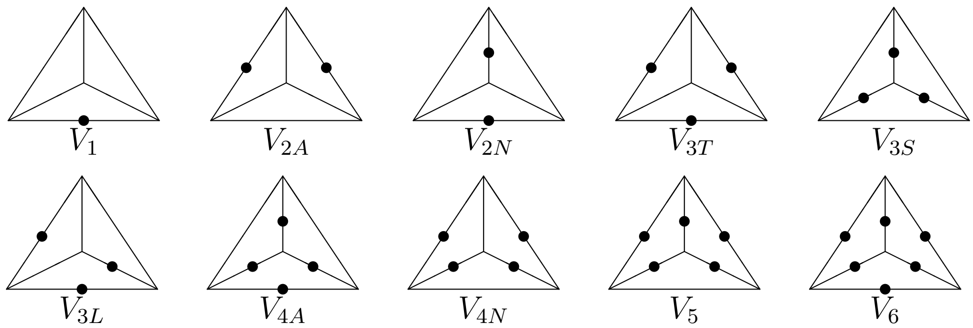


Fig 2: Colourings of the tetrahedron by mass (denoted by a blob)



Defining the finite two-point function (now with space-like p^2)

$$I(r_1 \dots r_5; p^2/m^2) := \frac{p^2}{\pi^4} \int d^4k \int d^4l \quad P_1(k) P_2(p+k) P_3(k-l) P_4(l) P_5(p+l) \tag{5}$$

in 4 dimensions, we obtain

$$V(r_1 \dots r_5, 0) - V(r_1 \dots r_5, 1) = \int_0^\infty dx\, I(r_1 \dots r_5; x) \left\{ \frac{1}{x} - \frac{1}{x+1} \right\} + O(\varepsilon) \tag{6}$$

for the difference of vacuum diagrams with a massless and massive sixth propagator. Suppressing the parameters $r_1 \dots r_5$, temporarily, we exploit the dispersion relation

$$I(x) = \int_{s_0}^\infty ds\, \sigma(s) \left\{ \frac{1}{s+x} - \frac{1}{s} \right\} \tag{7}$$

where $-2\pi i \sigma(s) = I(-s+i0) - I(-s-i0)$ is the discontinuity across the cut $[-\infty, -s_0]$ on the negative axis. Integration by parts then gives

$$I(x) = \int_{s_0}^\infty ds\, \sigma'(s) \left\{ -\log\left(1 + \frac{x}{s}\right) + \log\left(1 + \frac{x}{s_0}\right) \right\} \tag{8}$$

where the constant term in the logarithmic weight function may be dropped if $\sigma(s_0) = 0$, as occurs when $s_0 = 0$. As $x \to \infty$, we obtain the universal asymptotic value

$$6\zeta(3) = I(\infty) = \int_{s_0}^\infty ds\, \sigma'(s) \left\{ \log(s) - \log(s_0) \right\} \tag{9}$$

with the $\log(s_0)$ term dropped when $s_0 = 0$. The finite difference in (6) is obtained from (8) as

$$\int_0^\infty dx\, I(x) \left\{ \frac{1}{x} - \frac{1}{x+1} \right\} = \int_{s_0}^\infty ds\, \sigma'(s) \left\{ L_2(s) - L_2(s_0) \right\} \tag{10}$$

with a dilogarithmic weight function

$$L_2(s) := \int_0^\infty \frac{dx}{x(x+1)} \log\left(\frac{1+x}{1+x/s}\right) = \mathrm{Li}_2(1-1/s) = -\tfrac{1}{2}\log^2(s) - \mathrm{Li}_2(1-s) \qquad (11)$$

that is chosen to satisfy $L_2(1) = 1$, thus enabling one to drop $L_2(s_0)$ for $s_0 = 0$ and $s_0 = 1$, which covers all the cases with $N \leq 3$ massive particles in the two-point function, and hence $N + 1 \leq 4$ massive particles in vacuum diagrams.

We now prove that the two terms in the weight function (11) can be separated to yield the finite parts of the vacuum diagrams combined in (6), as follows:

$$F(r_1 \ldots r_5, 0) = \tfrac{1}{2}\int_{s_0}^\infty ds\, \sigma'(r_1 \ldots r_5; s) \left\{\log^2(s) - \log^2(s_0)\right\} \qquad (12)$$

$$F(r_1 \ldots r_5, 1) = -\int_{s_0}^\infty ds\, \sigma'(r_1 \ldots r_5; s)\ \{\mathrm{Li}_2(1-s) - \mathrm{Li}_2(1-s_0)\} \qquad (13)$$

with constant terms in the weight functions that are inert when $s_0 = 0$ and when $s_0 = 1$. The proof uses the representation

$$I(x) = 6\zeta(3) + \int_{s_0}^\infty ds\, \sigma'(s)\, \{-\log(x+s) + \log(x+s_0)\} \qquad (14)$$

in which the asymptotic value (9) is subtracted. Then one obtains

$$\int_0^\infty dx\, \frac{I(\infty) - I(x)}{x+1} = -\int_{s_0}^\infty ds\, \sigma'(s)\, \{\mathrm{Li}_2(1-s) - \mathrm{Li}_2(1-s_0)\} \qquad (15)$$

There are 13 distinct cases of the two-point function (5) with at least one massive particle. We denote the spectral densities by

$$
\begin{aligned}
\sigma_1(s) &:= \sigma(1,0,0,0,0;s)\\
\sigma_3(s) &:= \sigma(0,0,1,0,0;s)\\
\sigma_{12}(s) &:= \sigma(1,1,0,0,0;s)\\
\sigma_{13}(s) &:= \sigma(1,0,1,0,0;s)\\
\sigma_{14}(s) &:= \sigma(1,0,0,1,0;s)\\
\sigma_{15}(s) &:= \sigma(1,0,0,0,1;s)\\
\overline{\sigma}_{12}(s) &:= \sigma(0,0,1,1,1;s)\\
\overline{\sigma}_{13}(s) &:= \sigma(0,1,0,1,1;s)\\
\overline{\sigma}_{14}(s) &:= \sigma(0,1,1,0,1;s)\\
\overline{\sigma}_{15}(s) &:= \sigma(0,1,1,1,0;s)\\
\overline{\sigma}_{1}(s) &:= \sigma(0,1,1,1,1;s)\\
\overline{\sigma}_{3}(s) &:= \sigma(1,1,0,1,1;s)\\
\overline{\sigma}(s) &:= \sigma(1,1,1,1,1;s)
\end{aligned}
$$

where the subscripts of $\sigma_{\rm massive}$ indicate the massive lines, while those of $\overline{\sigma}_{\rm massless}$ show the massless lines.

Since only $\overline{\sigma}'_{15}$, $\overline{\sigma}'_1$ and $\overline{\sigma}'$ involve an elliptic integral, there is a systematic polylog route to all finite parts, save that of the totally massive case, V_6.

Table 1: Integral to diagram dictionary

density	σ_1	σ_3	σ_{12}	σ_{13}	σ_{14}	σ_{15}	$\overline{\sigma}_{12}$	$\overline{\sigma}_{13}$	$\overline{\sigma}_{14}$	$\overline{\sigma}_{15}$	$\overline{\sigma}_1$	$\overline{\sigma}_3$	$\overline{\sigma}$
integral (12)	V_1	V_1	V_{2A}	V_{2A}	V_{2A}	V_{2N}	V_{3T}	V_{3L}	V_{3S}	V_{3L}	V_{4A}	V_{4N}	V_5
integral (13)	V_{2A}	V_{2N}	V_{3S}	V_{3L}	V_{3T}	V_{3L}	V_{4A}	V_{4A}	V_{4A}	V_{4N}	V_5	V_5	V_6

Specializing the analysis I to cases with $r_j^2 = r_j$, we obtain

$$\begin{aligned}
\sigma'(r_1 \ldots r_5; s) &= \left\{\sigma'_a(r_1 \ldots r_5; s)\,\Theta\left(s - (r_1 + r_2)^2\right) + (1 \leftrightarrow 4, 2 \leftrightarrow 5)\right\} \\
&+ \left\{\sigma'_b(r_1 \ldots r_5; s)\,\Theta\left(s - (r_2 + r_3 + r_4)^2\right) + (1 \leftrightarrow 2, 4 \leftrightarrow 5)\right\} \\
\sigma'_a(r_1 \ldots r_5; s) &:= 2\,\Re \int_{(r_4+r_5)^2}^{\infty} dx\, \frac{T(x, r_1, r_2, r_3, r_4, r_5)}{\Delta(s, r_1, r_2)} \frac{\partial}{\partial x} \left(\frac{\Delta(x, r_1, r_2)}{x - s + i0}\right) \qquad (16) \\
\sigma'_b(r_1 \ldots r_5; s) &:= 2\,\Re \int_{(r_3+r_4)^2}^{(\sqrt{s}-r_2)^2} dx\, \frac{\partial}{\partial s} \left(\frac{T(x, s, r_2, r_5, r_4, r_3)}{x - r_1 + i0}\right) \qquad (17) \\
T(s, a, b, c, d, e) &:= \operatorname{arctanh}\left(\frac{\Delta(s, a, b)\Delta(s, d, e)}{x^2 - x(a + b - 2c + d + e) + (a - b)(d - e)}\right) \qquad (18) \\
\Delta(a, b, c) &:= \sqrt{a^2 + b^2 + c^2 - 2ab - 2bc - 2ca} \qquad (19)
\end{aligned}$$

with integration by parts in (16) giving a logarithmic result, in all cases, and differentiation in (17) giving a logarithmic result when $r_1 r_3 r_5 = r_2 r_3 r_4 = 0$, i.e. when there is no intermediate state with 3 massive particles.

3 The totally massive case

We were able to hand 9 cases by methods that avoided intermediate states with 3 massive particles. Now there is no option, since

$$F_6 = -\int_4^\infty ds\,\overline{\sigma}'(s)\,\mathrm{Li}_2(1-s) \tag{20}$$

involves intermediate states with two and three massive particles in

$$\overline{\sigma}'(s) = \overline{\sigma}'_a(s)\,\Theta(s-4) + \overline{\sigma}'_b(s)\,\Theta(s-9)\,. \tag{21}$$

We may, however, simplify matters by separating these contributions in

$$F_6 - F_5 \;=\; \int_4^\infty ds\,\overline{\sigma}'(s)\,\mathrm{Li}_2(1-1/s) = F_a + F_b \tag{22}$$

$$F_a \;:=\; \int_4^\infty ds\,\overline{\sigma}'_a(s)\;\{\mathrm{Li}_2(1-1/s) - \zeta(2)\} \tag{23}$$

$$F_b \;:=\; \int_9^\infty ds\,\overline{\sigma}'_b(s)\;\{\mathrm{Li}_2(1-1/s) - \zeta(2)\} \tag{24}$$

where F_5 may be evaluated without encountering elliptic integrals.

The two-particle cut gives a logarithm in

$$\overline{\sigma}'_a(s) = \frac{2}{s-3}\left\{\mathrm{arccosh}(s/2-1) - \frac{2\pi}{\sqrt{3s(s-4)}}\right\} \tag{25}$$

while the three-particle cut gives the elliptic[1] integral

$$\overline{\sigma}'_b(s) = -2\int_4^{(\sqrt{s}-1)^2} \frac{dx}{x-1}\,\frac{\Delta(x,1,1)}{\Delta(x,s,1)}\,\frac{x+s-1}{\Delta^2(x,s,1)+xs}\,. \tag{26}$$

At large s, contributions (25,26) are each $O(\log(s)/s)$, while their sum is $O(\log(s)/s^2)$. The integrals (23,24) converge separately, thanks to the $\zeta(2)$ in their weight functions, to which the combination (22) is blind.

It appears that we need to integrate the product of a dilog and an elliptic integral. To avoid this, we may we reverse the order of integration Setting $x = 1/u^2 \in [4,\infty]$ in (26), which now becomes the outer integration, and $s = (1/u+v)(1/u+1/v) \in [(1/u+1)^2,\infty]$ in the inner, we then integrate by parts on $v \in [0,1]$ to convert the dilog to a product of logs, with the result

$$F_b \;=\; 2\int_0^{\frac{1}{2}} du\left(\frac{dA(u)}{du}\right)\int_0^1 dv\left(\frac{\partial B(u,v)}{\partial v}\right) C(u,v)\,D(u,v) \tag{27}$$

$$A(u) \;:=\; \log\left(\frac{u^2}{1-u^2}\right) \tag{28}$$

$$B(u,v) \;:=\; \log\left(\frac{(1+uv)(u+v)}{u+v+uv^2}\right) \tag{29}$$

$$C(u,v) \;:=\; \log\left(\frac{(1+uv)(u+v)}{u^2v}\right) \tag{30}$$

$$D(u,v) \;:=\; \log\left(\frac{1+2uv+v^2+(1-v^2)\sqrt{1-4u^2}}{1+2uv+v^2-(1-v^2)\sqrt{1-4u^2}}\right) \tag{31}$$

[1] I am told that Källén was disappointed to find that the two-loop electron propagator involves an elliptic integral, unlike the simpler photon propagator.

which establishes that F_b is the integral of a trilogarithm.

The NAG routine D01FCF is notably efficient at evaluating rectangular double integrals in double-precision FORTRAN, which was ample to to discover the remarkable relation

$$F_6 = F_{3S} + F_{4N} - F_{2N} = 4\left(\mathrm{Cl}_2^2(\pi/3) + 4\zeta(4) + 2\zeta(\overline{3},\overline{1})\right) \tag{32}$$

This corresponds to a direct relation between diagrams

$$V_6 + V_{2N} = V_{3S} + V_{4N} + O(\varepsilon) \tag{33}$$

verified to 15 digits. It stands as testament to the oft remarked fact that results in quantum field theory have a simplicity that tends to increase with the labour expended.

4 Massive bananas

4.1 Schwinger's bananas

Let A be the diagonal $N \times N$ matrix with entries $A_{i,j} = \delta_{i,j}\alpha_i$. Let U be the column vector of length N with unit entries, $U_i = 1$. Then $B = U\widetilde{U}$ is the $N \times N$ matrix with unit entries, $B_{i,j} = 1$. The banana diagram with $N+1$ edges of unit mass, in two space-time dimensions, may be evaluated by Schwinger's trick as a multiple of the N-dimensional integral

$$\overline{V}_{N+1} = \int_{\alpha_i>0} \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N}{\mathrm{Det}(A+B)(\mathrm{Tr}(A)+1)} \tag{34}$$

where

$$\mathrm{Det}(A+B) = \sum_{i=0}^{N} \frac{1}{\alpha_i} \prod_{j=0}^{N} \alpha_j$$

is the first Symanzik polynomial, with $\alpha_0 = 1$ fixed by momentum conservation, and the second Symanzik polynomial

$$\mathrm{Tr}(A) + 1 = \sum_{i=0}^{N} \alpha_i$$

results from the fact that the $N+1$ edges are propagators with unit mass.

4.2 Bessels's bananas

We may also evaluate banana diagrams in x-space, since the two-dimensional Fourier transform of the p-space Euclidean propagator $1/(p^2+m^2)$, with $p^2 = p_0^2 + p_1^2$, yields the

Bessel function $K_0(mx)$, with $x^2 = x_0^2 + x_1^2$. The normalization in (34) corresponds to

$$\overline{V}_{N+1} = 2^N \int_0^\infty [K_0(t)]^{N+1} t \, \mathrm{d}t \tag{35}$$

which differs by a power of 2 from the Bessel moments that I studied with Bailey, Borwein and Glasser [1].

Hence I but a bar over V and use the subscript $N+1$ to indicate the number of Bessel functions.

4.3 Known bananas

It is proven that [1]

$$\begin{aligned} \overline{V}_1 &= 1 && (36)\\ \overline{V}_2 &= 1 && (37)\\ \overline{V}_3 &= 3L_{-3}(2) && (38)\\ \overline{V}_4 &= 7\zeta(3) && (39) \end{aligned}$$

where

$$L_{-3}(s) = \sum_{n\geq 0} \left(\frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s} \right)$$

is the Dirichlet L function with conductor -3.

The zero-loop evaluation (36) merely checks our normalization.

The one-loop evaluation

$$\overline{V}_2 = \int_0^\infty \frac{\mathrm{d}\alpha_1}{(\alpha_1+1)^2} = 1$$

follows neatly from (34), since with $N=1$ we have $\mathrm{Det}(A+B) = \mathrm{Tr}(A)+1 = \alpha_1+1$. I shall now use the letters $\{a,\, b,\, c,\, \ldots\}$ for the Schwinger parameters $\{\alpha_1,\, \alpha_2,\, \alpha_3,\, \ldots\}$.

4.4 Three-edge banana and sixth root of unity

At two loops, the Schwinger method gives the banana diagram with 3 edges as

$$\overline{V}_3 = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{(ab+a+b)(a+b+1)}.$$

To proceed we may take partial fractions with respect to b. Then

$$\frac{a^2+a+1}{(ab+a+b)(a+b+1)} = \frac{a+1}{ab+a+b} - \frac{1}{a+b+1} = \frac{\partial}{\partial b}\log\left(\frac{ab+a+b}{a+b+1}\right)$$

enables integration over b. Hence we obtain

$$\overline{V}_3 = \int_0^\infty \frac{G(a)\,\mathrm{d}a}{a^2+a+1} \tag{40}$$

with contributions to

$$G(a) = \log(1+a) + \log(1+1/a) \tag{41}$$

at $b=\infty$ and $b=0$. It is apparent from (40) that the sixth root of unity $\lambda=(1+\mathrm{i}\sqrt{3})/2$ is implicated, since $a^2+a+1=(a+\lambda)(a+\overline{\lambda})$, where $\overline{\lambda}=(1-\mathrm{i}\sqrt{3})/2=1-\lambda$ is the conjugate root. Working out the corresponding dilogarithms we obtain

$$\overline{V}_3=\frac{4}{\sqrt{3}}\Im\mathrm{Li}_2(\lambda)=3L_{-3}(2)$$

in agreement with the well known result (38).

4.5 Four-edge banana and $\zeta(3)$

To evaluate

$$\overline{V}_4=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(abc+ab+bc+ca)(a+b+c+1)}$$

we take partial fractions with respect to c and then integrate over c, to obtain

$$\overline{V}_4=\int_0^\infty\int_0^\infty\frac{L(a,b)\,\mathrm{d}a\,\mathrm{d}b}{(a+1)(b+1)(a+b)}$$

with

$$L(a,b)=\log\left(\frac{(ab+a+b)(a+b+1)}{ab}\right).$$

Hence with

$$F(a)=\int_0^\infty\frac{(a-1)L(a,b)\,\mathrm{d}b}{(b+1)(a+b)}$$

we have

$$\overline{V}_4=\int_0^\infty\frac{F(a)\,\mathrm{d}a}{a^2-1}=\int_0^1\frac{(F(a)-F(1/a))\,\mathrm{d}a}{a^2-1}. \tag{42}$$

I shall need only the derivative of $F(a)$. Let

$$K(a,b) = \frac{b\,L(a,b)}{a+b} + \log(ab+a+b) - 2\log(a+b+1).$$

Then, by construction,

$$\frac{\partial}{\partial b}K(a,b) = a\frac{\partial}{\partial a}\left(\frac{(a-1)L(a,b)}{(b+1)(a+b)}\right)$$

and hence

$$a\frac{\mathrm{d}}{\mathrm{d}a}F(a) = K(a,\infty) - K(a,0) = 2G(a)$$

where $G(a)$ was given in (41). We now integrate (42) by parts, to obtain

$$\overline{V}_4 = \int_0^1 \frac{\mathrm{d}a}{a}\,\log\left(\frac{1+a}{1-a}\right)(G(a)+G(1/a))$$

and use Nielsen's evaluations

$$\begin{aligned}
-\int_0^1 \frac{\mathrm{d}a}{a}\,\log(1-a)\log(1+a) &= \frac{5}{8}\zeta(3)\\
-\int_0^1 \frac{\mathrm{d}a}{a}\,\log(a)\log(1+a) &= \frac{3}{4}\zeta(3)\\
\int_0^1 \frac{\mathrm{d}a}{a}\,\log^2(1+a) &= \frac{1}{4}\zeta(3)\\
\int_0^1 \frac{\mathrm{d}a}{a}\,\log(a)\log(1-a) &= \zeta(3)
\end{aligned}$$

to obtain

$$\overline{V}_4 = \left(4\times\frac{5}{8} + 2\times\frac{3}{4} + 4\times\frac{1}{4} + 2\right)\zeta(3) = 7\zeta(3)$$

in agreement with the previously known result (39).

4.6 Unknown banana

The next diagram has 5 edges and hence 4 loops. After an easy first integration, we obtain

$$\overline{V}_5 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{M(a,b,c)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(ab+a+b)c^2 + (ab+a+b)(a+b)c + (a+b)ab}$$

with

$$M(a,b,c) = \log(a+b+c+1) + \log\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

But then integration over c will produce complicated dilogarithms with arguments involving the square root of the discriminant

$$D(a,b) = (ab+a+b)(a+b)(ab(a+b) + (a-b)^2)$$

of the quadratic in c. The result will have the form

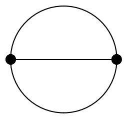
$$\overline{V}_5 = \int_0^\infty \int_0^\infty \frac{L_2(a,b)\mathrm{d}a\,\mathrm{d}b}{\sqrt{D(a,b)}}$$

with undisclosed dilogs in $L_2(a,b)$. Integration by parts, to reduce the dilogs to logs, would require us to introduce an elliptic function, since $D(a,b)$ is a quartic in b.

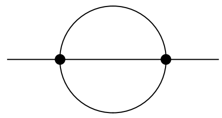
We know nothing about the number theory of $\overline{V}_5$. Its value is known to 1000 decimal places.

5 Cut bananas

For $N > 2$ we may cut an edge in $\overline{V}_N$ and set the two external half edges on the unit mass shell, which is at $p^2 = -1$. I call the result $\overline{S}_N$. It has $N - 1$ internal edges and hence $N - 2$ loops. Thus $\overline{V}_3$ and $\overline{S}_4$ correspond to the two-loop diagrams



$\overline{V}_3$



$\overline{S}_4$

with the "sunrise" diagram $\overline{S}_4$ obtained by cutting an edge of $\overline{V}_4$.

5.1 Schwinger's cut bananas

At N loops, the integral over Schwinger parameters is

$$\overline{S}_{N+2} = \int_{\alpha_i > 0} \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N}{\mathrm{Det}(A + B)\mathrm{Tr}(A) + \widetilde{U}CU}. \tag{43}$$

where C is the adjoint of $A + B$, with

$$(A + B)C = \mathrm{Det}(A + B)I$$

where I is the unit matrix with $I_{i,j} = \delta_{i,j}$. The denominator in (43) is the second Symanzik polynomial.

5.2 Bessels's cut bananas

In x-space, cutting an edge and putting it on the mass shell corresponds to replacing one instance of the Bessel function $K_0(t)$ by $I_0(t)$, to obtain

$$\overline{S}_{N+2} = 2^N \int_0^\infty I_0(t)[K_0(t)]^{N+1} t \, \mathrm{d}t \tag{44}$$

at N loops. Note that $\overline{S}_2$ is divergent, since

$$I_0(t) = \sum_{k \geq 0} \left(\frac{t^k}{2^k k!} \right)^2$$

grows exponentially, with

$$I_0(t) = \frac{\exp(t)}{\sqrt{2\pi t}} \left(1 + \frac{1}{8t} + O(1/t^2) \right)$$

as $t \to \infty$, while

$$K_0(t) = \sqrt{\frac{\pi}{2t}} \exp(-t) \left(1 - \frac{1}{8t} + O(1/t^2) \right)$$

is exponentially damped.

5.3 Known cut bananas

It is proven that [1]

$$\overline{S}_3 \;=\; 2L_{-3}(1) = \frac{2\pi}{3\sqrt{3}} \tag{45}$$

$$\overline{S}_4 \quad = \quad \mathrm{Li}_2(1) - \mathrm{Li}_2(-1) = \frac{\pi^2}{4} \tag{46}$$

and it is conjectured that [1]

$$\overline{S}_5 \stackrel{?}{=} \frac{1}{30\sqrt{5}}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) \tag{47}$$

which holds to at least 1000 decimal places.

5.4 Cut banana with sixth root of unity

The Schwinger formula (43) at one loop gives

$$\overline{S}_3 = \int_0^\infty \frac{\mathrm{d}a}{a^2+a+1} = \frac{\log(\lambda)-\log(\overline{\lambda})}{\lambda-\overline{\lambda}} = \frac{2\arctan(\sqrt{3})}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$$

as claimed in (45).

5.5 Cut banana with π^2

At two loops, we have

$$\overline{S}_4 = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{(a+b)(a+1)(b+1)}$$

with a convenient factorization of the second Symanzik polynomial. Hence

$$\overline{S}_4 = \int_0^\infty \frac{\log(a)\,\mathrm{d}a}{a^2-1} = 2\int_0^1 \frac{\log(a)\,\mathrm{d}a}{a^2-1}$$

yields dilogs at square roots of unity, namely

$$\overline{S}_4 = \mathrm{Li}_2(1) - \mathrm{Li}_2(-1) = \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{\pi^2}{4}$$

as claimed in (46).

5.6 Cut banana at the 15th singular value

At three loops, we have

$$\overline{S}_5 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{P(a,b,c)}$$

where

$$P(a,b,c) = (abc + ab + bc + ca)(a + b + c) + (ab + bc + ca)$$

with the final term, $(ab + bc + ca)$, resulting from the adjoint matrix. Grouping powers of c, we see that

$$P(a,b,c) = (ab + a + b)c^2 + (ab + a + b)(a + b + 1)c + (a + b + 1)ab$$

yields a discriminant

$$\Delta(a,b) = (ab + a + b)(a + b + 1)((ab + a + b)(a + b + 1) - 4ab)$$

and the integral over c gives

$$\overline{S}_5 = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{\Delta(a,b)}} \log\left(\frac{1 + X(a,b)}{1 - X(a,b)}\right)$$

with

$$X(a,b) = \sqrt{1 - \frac{4ab}{(ab+a+b)(a+b+1)}}\,.$$

Conjecture (47) was stimulated by a proven result for

$$\overline{T}_5 \equiv 4\int_0^\infty [I_0(t)]^2[K_0(t)]^3 t\,\mathrm{d}t = \int_0^\infty\int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{\Delta(a,b)}}$$

namely

$$\overline{T}_5 = \frac{\sqrt{3}}{120\pi}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) \tag{48}$$

obtained at the 15th singular value, by diamond mining [1].

Numerical evaluation suggested that

$$\frac{\overline{S}_5}{\overline{T}_5} \stackrel{?}{=} \frac{4\pi}{\sqrt{15}}$$

and this has been confirmed at 1000-digit precision. Yet it remains to be proved that

$$\int_0^\infty\int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{\Delta(a,b)}}\left(\log\left(\frac{1+X(a,b)}{1-X(a,b)}\right) - \frac{4\pi}{\sqrt{15}}\right)$$

vanishes. It has been shown that its magnitude is smaller than 10^{-1000}.

6 L-series of a K3 surface

For $s > 2$ let

$$L(s) = \prod_p \frac{1}{1 - \frac{A_p}{p^s} + \left(\frac{p}{15}\right)\frac{p^2}{p^{2s}}}$$

where $\left(\frac{\cdot}{15}\right)$ is a Kronecker symbol and the product is over all primes p, with integers

$$\begin{aligned} A_3 &= -3, \\ A_5 &= 5, \\ A_p &= 0, \text{ for } \left(\frac{p}{15}\right) = -1, \\ A_p &= 2x^2 + 2xy - 7y^2, \text{ for } x^2 + xy + 4y^2 = p \equiv 1, 4 \pmod{15}, && (49) \\ A_p &= x^2 + 8xy + y^2, \text{ for } 2x^2 + xy + 2y^2 = p \equiv 2, 8 \pmod{15}, && (50) \end{aligned}$$

with pairs of integers (x, y) defined, for $x > 0$, by the quadratic forms in (49,50).

As shown by Peters, Top and van der Vlugt [2], the L-series

$$L(s) = \sum_{n>0} \frac{A_n}{n^s}$$

is generated by the weight-3 modular form

$$f_3(q) = \eta(q)\eta(q^3)\eta(q^5)\eta(q^{15})R(q) = \sum_{n>0} A_n q^n \qquad (51)$$

where

$$\frac{\eta(q)}{q^{1/24}} = \prod_{j>0}(1-q^j) = \sum_{n\in\mathbf{Z}}(-1)^n q^{n(3n+1)/2}, \quad (52)$$

$$R(q) = \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+4n^2}. \quad (53)$$

Note that $A_1 = 1$, since $1+3+5+15 = 24$. If $q = p^r$ is a prime power, then

$$A_{pq} = A_p A_q - \left(\frac{p}{15}\right) p^2 A_{q/p}.$$

If $n = \prod_j q_j$, with prime powers $q_j = p_j^{r_j}$, then $A_n = \prod_j A_{q_j}$. Thus (49,50) suffice to compute A_n and are easily programmed using the `qfbsolve` command of `Pari-GP`.

I now describe how I was able to evaluate 20000 good digits of the conditionally convergent series $L(2) = \sum_{n>0} A_n/n^2$. Let

$$\Lambda(s) = \frac{\Gamma(s)}{c^s}L(s), \text{ with } c = \frac{2\pi}{\sqrt{15}}.$$

Then [2] gives the functional equation $\Lambda(s) = \Lambda(3-s)$, which may be used to extend the Mellin transform

$$\Lambda(s) = \sum_{n>0} A_n \int_0^\infty \frac{\mathrm{d}x}{x} x^s \exp(-cnx) \quad (54)$$

throughout the complex s-plane, as follows

$$\Lambda(s) = \sum_{n>0} A_n \left(\frac{\Gamma(s, cn\lambda)}{(cn)^s} + \frac{\Gamma(3-s, cn/\lambda)}{(cn)^{3-s}}\right) \quad (55)$$

where

$$\Gamma(s,y) = \int_y^\infty \frac{\mathrm{d}x}{x} x^s \exp(-x)$$

is the incomplete Γ function and $\lambda \geq 0$ is an arbitrary real parameter. To establish (55), I remark that it agrees with (54), at $\lambda = 0$, and that its derivative with respect to λ vanishes by virtue of the inversion symmetry

$$M(\lambda) \equiv \lambda^{3/2} \sum_{n>0} A_n \exp(-cn\lambda) = M(1/\lambda).$$

Optimal convergence is achieved at $\lambda = 1$, where

$$\Lambda(s) = \sum_{n>0} A_n \int_1^\infty \frac{\mathrm{d}x}{x} \left(x^s + x^{3-s}\right) \exp\left(-\frac{2\pi n x}{\sqrt{15}}\right) \tag{56}$$

makes the relation $\Lambda(s) = \Lambda(3-s)$ explicit. Zeros on the critical line $\Re s = 3/2$ occur when

$$\Lambda(3/2 + \mathrm{i}s_0) = 2 \sum_{n>0} A_n \int_1^\infty \mathrm{d}x\, x^{1/2} \cos(s_0 \log(x)) \exp\left(-\frac{2\pi n x}{\sqrt{15}}\right)$$

vanishes. I have computed 100 good digits of the first zero, obtaining

$$\begin{aligned} s_0 \quad = \quad & 4.841925814229962588045533711247175448399945840634 7\\ & 669395095360856334816804741135372158525188377525005\ldots \end{aligned}$$

At $s = 2$, the integral in (56) is elementary and we have dramatically improved convergence for

$$L(2) \equiv \sum_{n>0} \frac{A_n}{n^2} = \sum_{n>0} \frac{A_n}{n^2} \left(1 + \frac{4\pi n}{\sqrt{15}}\right) \exp\left(-\frac{2\pi n}{\sqrt{15}}\right) \tag{57}$$

from which I obtained more than 20000 good digits in less than a minute, by computing the first 30000 terms, with the aid of (49,50). The result is consistent with the conjecture

$$\begin{aligned}
3L(2) &\overset{?}{=} \overline{T}_5 && (58)\\
&\equiv 4\int_0^\infty [I_0(t)]^2[K_0(t)]^3 t\,\mathrm{d}t && (59)\\
&= \int_0^\infty\int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{(ab+a+b)(a+b+1)((ab+a+b)(a+b+1)-4ab)}} && (60)\\
&= \frac{\pi^2}{8}\left(\sqrt{15}-\sqrt{3}\right)\left(1+2\sum_{n>0}\exp\left(-\sqrt{15}\pi n^2\right)\right)^4 && (61)\\
&= \frac{\sqrt{3}}{120\pi}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) && (62)\\
&\overset{?}{=} \frac{\sqrt{15}}{4\pi}\overline{S}_5 && (63)
\end{aligned}$$

where $\overline{T}_5$ is defined in (59) as a Bessel moment, with a proven integral representation over Schwinger parameters in (60), a proven evaluation at the 15th singular value in (61), a proven reduction to Γ values in (62) and a conjectural relation to $\overline{S}_5$ in (63).

Unfortunately, I did not succeed in relating $\overline{V}_5$ to $L(3)$ and/or $L(4)$.

7 L-series for 6 Bessel functions

We are interested in relating Bessel moments of the form

$$\overline{V}_N = 2^{N-1}\int_0^\infty [K_0(t)]^N t\,\mathrm{d}t, \text{ for } N>0, \tag{64}$$

$$\overline{S}_N = 2^{N-2}\int_0^\infty I_0(t)[K_0(t)]^{N-1} t\,\mathrm{d}t, \text{ for } N>2, \tag{65}$$

$$\overline{T}_N = 2^{N-3}\int_0^\infty I_0^2(t)[K_0(t)]^{N-2} t\,\mathrm{d}t, \text{ for } N>4, \tag{66}$$

$$\overline{U}_N = 2^{N-4}\int_0^\infty I_0^3(t)[K_0(t)]^{N-3} t\,\mathrm{d}t, \text{ for } N\geq 6, \tag{67}$$

$$\overline{W}_N = 2^{N-5}\int_0^\infty I_0^4(t)[K_0(t)]^{N-4} t\,\mathrm{d}t, \text{ for } N\geq 8, \tag{68}$$

to L-series derived from modular forms. In [1] it was conjectured that

$$\overline{S}_5 \stackrel{?}{=} \frac{4\pi}{\sqrt{15}}\,\overline{T}_5 \tag{69}$$

$$\overline{S}_6 \stackrel{?}{=} \frac{4\pi^2}{3}\,\overline{U}_6 \tag{70}$$

$$\overline{T}_8 \stackrel{?}{=} \frac{18\pi^2}{7}\,\overline{W}_8 \tag{71}$$

with a notable appearance of 7 in the denominator on the right hand side of (71). Francis Brown suggested that the weight-4 modular form

$$f_4(q) = \left[\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)\right]^2 = \sum_{n>0} A_{4,n}q^n \tag{72}$$

of Hulek, Spandaw, van Geemen and van Straten [3] might yield an L-series

$$L_4(s)=\sum_{n>0}\frac{A_{4,n}}{n^s}=\frac{1}{1+2^{1-s}}\,\frac{1}{1+3^{1-s}}\prod_{p>3}\frac{1}{1-\frac{A_{4,p}}{p^s}+\frac{p^3}{p^{2s}}}$$

with values related to the problem with 6 Bessel functions. Note that $A_{4,1}=1$, since $2(1+2+3+6)=24$.

The Mellin transform

$$\Lambda_4(s)=\frac{\Gamma(s)}{(2\pi/\sqrt{6})^s}\,L_4(s)=\sum_{n>0}A_{4,n}\int_0^\infty\frac{\mathrm{d}x}{x}x^s\exp\left(-\frac{2\pi nx}{\sqrt{6}}\right)$$

may be analytically continued to give

$$\Lambda_4(s)=\sum_{n>0}A_{4,n}\int_1^\infty\frac{\mathrm{d}x}{x}\left(x^s+x^{4-s}\right)\exp\left(-\frac{2\pi nx}{\sqrt{6}}\right)$$

by virtue of the inversion symmetry

$$M_4(\lambda)\equiv\lambda^2\sum_{n>0}A_{4,n}\exp\left(-\frac{2\pi n\lambda}{\sqrt{6}}\right)=M_4(1/\lambda)$$

that gives the reflection symmetry $\Lambda_4(s)=\Lambda_4(4-s)$.

Then, at $s=2$ and $s=3$, we obtain the very convenient formulas

$$L_4(2)=\sum_{n>0}\frac{A_{4,n}}{n^2}\left(2+\frac{4\pi n}{\sqrt{6}}\right)\exp\left(-\frac{2\pi n}{\sqrt{6}}\right)\tag{73}$$

$$L_4(3)=\sum_{n>0}\frac{A_{4,n}}{n^3}\left(1+\frac{2\pi n}{\sqrt{6}}+\frac{2\pi^2n^2}{3}\right)\exp\left(-\frac{2\pi n}{\sqrt{6}}\right)\tag{74}$$

without resort to incomplete Γ functions that entail exponential integrals. By this means, I was able to compute 20000 good digits of (73,74) in less than 100 seconds. Then the conjectural evaluations

$$\overline{S}_6 \stackrel{?}{=} 48\zeta(2)L_4(2) \tag{75}$$

$$\overline{T}_6 \stackrel{?}{=} 12L_4(3) \tag{76}$$

$$\overline{U}_6 \stackrel{?}{=} 6L_4(2) \tag{77}$$

were discovered and checked at 1000-digit precision.

I remark that Francis Brown had expected a result of form (76), for $\overline{T}_6$, with an unknown rational coefficient, which I here evaluate as 12. The existence of a relation of the form (77), for $\overline{U}_6$, had not been predicted, since I had been unable to provide an expression for this Bessel moment as an integral over Schwinger parameters of an algebraic or polylogarithmic function. However, it was quite natural to guess that a reduction of $\overline{T}_6$ to $L_4(3)$ would be accompanied by a reduction of $\overline{U}_6$ to $L_4(2)$. Then the reduction of $\overline{S}_6$ to $\zeta(2)L_4(2)$ follows from conjecture (70), which I had already checked at 1000-digit precision in [1].

8 *L*-series for 8 Bessel functions

Next, Francis Brown provided the first 100 Fourier coefficients of a weight-6 modular form $f_6(q) = \sum_{n>0} A_{6,n}q^n$, whose *L*-series

$$L_6(s) = \sum_{n>0} \frac{A_{6,n}}{n^s} = \frac{1}{1-2^{2-s}} \frac{1}{1+3^{2-s}} \prod_{p>3} \frac{1}{1 - \frac{A_{6,p}}{p^s} + \frac{p^5}{p^{2s}}}$$

was expected to yield values related to the problem with 8 Bessel functions. His data may be condensed down to the values

`-66, 176, -60, -658, -414, 956, 600, 5574, -3592, -8458, 19194, 13316, -19680, -31266, 26340, -31090, -16804, 6120, -25558, 74408, -6468, -32742, 166082`

of $A_{6,p}$ for the primes $p = 5, 7, \ldots, 97$.

From this I inferred that the explicit modular form is given by

$$f_6(q) = g(q)g(q^2) \tag{78}$$

$$g(q) = \left[\eta(q)\eta(q^3)\right]^2 \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2} \tag{79}$$

with $f_6(q)/f_4(q)$ given by the θ function of the strongly 6-modular lattice [4] indexed by `http://www2.research.att.com/~njas/lattices/QQF.4.g.html` with expansion coefficients in entry `A125510` of Neil Sloane's On-Line Encyclopedia of Integer Sequences.

Proceeding along the lines of the previous section, I accelerated the convergence of

$$\Lambda_6(s) = \frac{\Gamma(s)}{(2\pi/\sqrt{6})^s} L_6(s) = \sum_{n>0} A_{6,n} \int_0^\infty \frac{\mathrm{d}x}{x} x^s \exp\left(-\frac{2\pi n x}{\sqrt{6}}\right)$$

by using the functional relation $\Lambda_6(s) = \Lambda_6(6-s)$ to obtain

$$\Lambda_6(s) = \sum_{n>0} A_{6,n} \int_1^\infty \frac{\mathrm{d}x}{x} \left(x^s + x^{6-s}\right) \exp\left(-\frac{2\pi n x}{\sqrt{6}}\right)$$

and hence the convenient formulas

$$L_6(3) = \sum_{n>0} \frac{A_{6,n}}{n^3}\left(2 + \frac{4\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right), \tag{80}$$

$$L_6(4) = \sum_{n>0} \frac{A_{6,n}}{n^4} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{4\pi^2 n^2}{9} + \frac{4\pi^3 n^3}{9\sqrt{6}}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right), \quad (81)$$

$$L_6(5) = \sum_{n>0} \frac{A_{6,n}}{n^5} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{\pi^2 n^2}{3} + \frac{2\pi^3 n^3}{9\sqrt{6}} + \frac{\pi^4 n^4}{27}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right). \quad (82)$$

The resulting fits

$$\overline{T}_8 \stackrel{?}{=} 216 L_6(5) \quad (83)$$

$$\overline{U}_8 \stackrel{?}{=} 36 L_6(4) \quad (84)$$

$$\overline{W}_8 \stackrel{?}{=} 8 L_6(3) \quad (85)$$

are rather satisfying. They leave the conjectural relation

$$L_6(5) \stackrel{?}{=} \frac{4}{7}\zeta(2) L_6(3) \quad (86)$$

as a restatement of the notable conjecture (71) given in [1].

Thanks to the explicit formula (78) for the weight-6 modular form, conjecture (86) has now been checked to 20000-digit precision.

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