Parametric Feynman Integrals & Hyperlogarithms

[Francis Brown: 0804.1660v1 , 0910.0114v2]

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Renormalized Feynman amplitudes and coefficients of the DimReg ε-Expansion are periods with parametric representations like

$$\left(\ln \frac{q^2}{\mu^2}\right) \int \frac{\Omega}{\psi^2}, \quad \int \frac{\Omega}{\psi^2} \ln \frac{\varphi}{\varphi^0}, \quad \int \Omega \left[\frac{\ln \frac{\varphi}{\varphi^0}}{\psi^2} - \frac{\ln \dfrac{\varphi_{\Gamma/\gamma} \psi_\gamma + \varphi^0_\gamma \psi_{\Gamma/\gamma}}{\varphi^0_{\Gamma/\gamma} \psi_\gamma + \varphi^0_\gamma \psi_{\Gamma/\gamma}}}{\psi_\gamma^2 \; \psi_{\Gamma/\gamma}^2} \right]$$

The known periods are mostly MZV, alternating sums, $\frac{1}{\sqrt{3}} \operatorname{Im} \operatorname{Li}_2(\xi_3), \ldots$

This simplicity originates from combinatorics of graph polynomials:

$$\int \frac{\Omega}{\psi^2} = \int \frac{\Omega}{\psi^1 \psi_1} = \int \frac{\Omega}{(\psi^{1,2})^2} \ln \frac{\psi_2^1 \psi_1^2}{\psi^{12} \psi_{12}}$$

$$= \int \Omega \left\{ \frac{\psi^{123} \ln \psi^{123}}{\psi^{12,13} \psi^{12,23} \psi^{13,23}} - \frac{\psi_{123} \ln \psi_{123}}{\psi_1^{2,3} \psi_2^{1,3} \psi_3^{1,2}} + \sum_{\substack{i=1 \\ \{j,k,i\}=\{1,2,3\}}}^{3} \left(\frac{\psi^i_{jk} \ln \psi^i_{jk}}{\psi^{ij,ik} \psi_i^{i,j} \psi_j^{i,k}} - \frac{\psi_i^{jk} \ln \psi_i^{jk}}{\psi_i^{j,k} \psi^{ij,jk} \psi^{ik,jk}} \right) \right\}$$

Example:

$$P\Big($$  $$\Big) = \int \Omega \left\{ \frac{(x+y)\ln(x+y)}{xy(xy+xz+yz)} + \frac{(x+z)\ln(x+z)}{xz(xy+xz+yz)} + \frac{(y+z)\ln(y+z)}{yz(xy+xz+yz)} - \frac{\ln(xy+xz+yz)}{xyz} \right\}$$

In this linear situation, the partial (parametric) Feynman integrals are "polylogarithmic" functions of the remaining Schwinger parameters. To identify these, we need a basis of such functions. Candidates are

- $\operatorname{Li}_{n_1,\ldots,n_r}(z_1,\ldots,z_r) = \sum_{0<k_1<\ldots<k_r} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}}$
- Iterated integrals in multiple variables as Christian Bogner will explain
- Hyperlogarithms, dissection polylogarithms

Hyperlogarithms

We allow singularities in $\Sigma = \{\overset{0}{\overset{\shortparallel}{\sigma_0}}, \ldots, \sigma_N\} \subset \mathbb{C}$ and introduce formal letters ω_σ forming the alphabet $A = \{\omega_\sigma \mid \sigma \in \Sigma\}$. The words $A^\times$ span the tensor algebra $T(A)$, on which

i) $L_{\omega_0^n}(z) := \frac{\log^n z}{n!}$ ii) $\partial_z L_{\omega_\sigma w}(z) = \frac{L_w(z)}{z-\sigma}$

iii) $\lim_{z \to 0} L_w(z) = 0 \quad \forall\, w \notin \{\omega_0^n \mid n \in \mathbb{N}_0\}$

uniquely determines a family of multivalued functions (on $\mathbb{C}\setminus\Sigma$)

$$L_{\omega_{\sigma_1}\ldots\omega_{\sigma_n}}(z) = \int_0^z \frac{dz_1}{z_1-\sigma_1} \int_0^{z_1} \frac{dz_2}{z_2-\sigma_2} \cdots \int_0^{z_{n-1}} \frac{dz_n}{z_n-\sigma_n} \qquad (\sigma_n \neq 0).$$

Examples: i) $L_{\omega_\sigma}(z) = \log \frac{z-\sigma}{-\sigma} = \log\left(1-\frac{z}{\sigma}\right) \quad \forall \sigma \neq 0$

ii) $\mathrm{Li}_n(z) = -L_{\omega_0^{n-1}\omega_1}(z)$

iii) $\mathrm{Li}_{n_1,\ldots,n_r}(z) = (-1)^r L_{\omega_0^{n_r-1}\omega_1 \ldots \omega_0^{n_1-1}\omega_1}(z)$

The shuffle-product $L_w(z) \cdot L_u(z) = L_{w \sqcup\!\sqcup u}(z)$ turns L into a character on $T_{\sqcup\!\sqcup}(A)$.

At any $\sigma \in \Sigma$, $L_w(z)$ grows at most logarithmically:

$$L_w(z) = \sum_k \log^k(z-\sigma) \cdot L_{w_k}(z) \qquad (*)$$

for $L_{w_k}(z)$ analytic at $z \to \sigma$ (uniquely determining w_k).

- $\{L_w(z) \mid w \in A^\times\}$ are linearly independent over $\mathbb{C}(z)$
- $\mathbb{C}(z) \otimes L(T(A))$ resp. $\mathbb{C}(z) \otimes T(A)$ is closed under $\partial/\partial z$ and $\int dz$
- Combinatorial algorithms exist for all these operations and decompositions $(*)$

Let $\sigma_1(t),\dots,\sigma_n(t)$ be functions of t and $\sigma_i' := \frac{\partial}{\partial t}\sigma_i$, then

$$\partial_{/\partial t} L_{\omega_{\sigma_1}\cdots\omega_{\sigma_n}}(z) = -\frac{\sigma_1'}{z-\sigma_1} L_{\omega_{\sigma_2}\cdots\omega_{\sigma_n}}(z)$$

$$+\sum_{k=1}^{n-1} \frac{(\sigma_i-\sigma_{i+1})'}{\sigma_i-\sigma_{i+1}}\left[L_{\dots\cancel{\omega_{\sigma_{i+1}}}\dots}(z) - L_{\dots\cancel{\omega_{\sigma_i}}\dots}(z)\right] - \frac{\sigma_n'}{\sigma_n} L_{\omega_{\sigma_1}\cdots\omega_{\sigma_{n-1}}}(z)$$

Hence, $L_w(\infty) = L_u(t)$ is an iterated integral!

Examples:

$$L_{(\omega_0-\omega_{-y/1+y})\,\omega_{-y/1+y}}(\infty) = \zeta(2)$$

$$L_{(\omega_0-\omega_{-y/1+y})\,\omega_{-y}}(\infty) = \zeta(2) + L_{(\omega_{-1}-\omega_0)\,\omega_{-1}}(y)$$

$$L_{(\omega_0-\omega_{-y/1+y})\,\omega_{-1}}(\infty) = L_{\omega_{-1}(\omega_{-1}-\omega_0)}(y)$$

Polynomial reduction

This algorithm depends on:

- linear denominators w.r.t. variable of integration
- complete factorization of $(\sigma_i-\sigma_j)(t)$:

$$\frac{(\sigma_i-\sigma_j)'}{\sigma_i-\sigma_j} = \frac{\partial}{\partial t}\ln(\sigma_i-\sigma_j) = \sum_\tau \frac{\lambda_\tau}{t-\tau}, \quad (\sigma_i-\sigma_j)(t) = \prod_\tau (t-\tau)^{\lambda_\tau}$$

We can determine all these polynomials and conclude

- if we can apply the algorithm at all
- which kind of period we will end up with

$$P = L_w(\infty),\quad w\in T(A),\quad \Sigma \subseteq \begin{cases} \{0,-1\} \\ \{1,0,-1\} \\ \{0\}\cup\{\xi \mid \xi^p=1\} \\ \dots \end{cases}$$

Examples

Consider the ε-expansion of massless propagators:

$\boxed{\text{2 loops}}$ 

$\Rightarrow$ MZV

$\boxed{\text{3 loops}}$ 

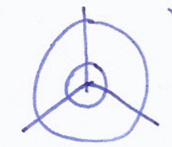
$\Rightarrow$ MZV

 $\Rightarrow$ alternating Euler sums (?)

$\boxed{\text{4 loops}}$ MZV's, possibly alternating sums in 

Comments, remarks & outlook

- efficient combinatorial algorithm giving exact results
- no integration-by-parts or reduction to masterintegrals needed
- proof-machine for Feynman periods (e.g. vertex-width ≤ 3)
- not limited to primitive graphs

$P_{BPHZ}\Big($  $\Big) = 96\,\mathrm{Li}_7(-1) + 128\,\mathrm{Li}_3^2(-1) = 72[\zeta(3)]^2 - \frac{189}{2}\zeta(7)$

- Gauge theory or tensor integrals are
  - i) not harder to calculate
  - ii) more difficult to predict in weight
- certain massive integrals are amenable to the algorithm
- multi-scale-amplitudes
- infrared divergences
- changing variables; Schwinger parameters not always optimal
- not restricted to 4 dimensions