A Quintic Hypersurface in $\mathbb{P}^8(\mathbb{C})$ with Many Nodes

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Abstract

We construct a hypersurface of degree 5 in projective space $\mathbb{P}^8(\mathbb{C})$ which contains exactly 23436 ordinary nodes and no further singularities. This limits the maximum number $\mu_8(5)$ of ordinary nodes a hyperquintic in $\mathbb{P}^8(\mathbb{C})$ can have to $23436 \leq \mu_8(5) \leq 27876$. Our method generalizes the approach by the 3^{rd} author for the construction of a quintic threefold with 130 nodes in an earlier paper.

Introduction

Let $\mu_n(d)$ be the maximum number of ordinary nodes a hypersurface of degree d in $\mathbb{P}^n := \mathbb{P}^n(\mathbb{C})$ can have. It is known only for a few nontrivial cases: For curves in the plane we have $\mu_2(d) = d(d-1)/2$. In three–space, $\mu_3(d)$ is only known for $d \leq 6$; see [Bar, JR] for the case of degree six and [Lab] for an extensive overview. In $\mathbb{P}^n$ with $n \geq 4$, the best known upper bound is Varchenko's *spectral bound* [Var]

$$\mu_n(d) \leq \mathrm{Ar}_n(d),$$

where $\mathrm{Ar}_n(d)$ is Arnold's number:

$$\mathrm{Ar}_n(d) := \#\Big\{(k_0,\dots,k_n) \in \big((0,d)\cap\mathbb{Z}\big)^{n+1} \;\Big|\; \sum_{i=0}^{n} k_i = \Big\lfloor \frac{nd}{2} \Big\rfloor + 1\Big\}.$$

All currently best known lower bounds follow from symmetric constructions: Kalker [Kal] constructed Σ_n-symmetric cubics which show $\mu_n(3) = \mathrm{Ar}_n(3) = \binom{n+1}{\lfloor \frac{n}{2} \rfloor}$ for any n. Goryunov constructed A_{n+1}- and B_{n+1}-symmetric quartics in $\mathbb{P}^n$, which reach approximately 86% of the Arnold-Varchenko upper bound (cf. [Gor]). In [vStr], a Σ_6-symmetric quintic in $\mathbb{P}^4$ with 130 nodes was

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constructed which limits the possibilities for $\mu_4(5)$ to $130 \leq \mu_4(5) \leq 135 = \mathrm{Ar}_4(5)$.
In sections 1 to 3, we consider the case of Σ_{n+2}-invariant quintics and construct an example in $\mathbb{P}^8$ with 23436 nodes which yields

$$23436 \leq \mu_8(5) \leq 27876 = \mathrm{Ar}_8(5).$$

For most other n, it seems that a pentagon–symmetric construction yields more nodes than our approach; we discuss this briefly in section 4.

1 Σ_{n+2}-symmetric Hyperquintics

Adapting the approach used in [vStr], we consider the 1-parameter-family of Σ_{n+2}-symmetric hyperquintics $Q := Q_{(\alpha:\beta)}$ given by

$$F_{(\alpha:\beta)} := \alpha S_5 + \beta S_2 S_3 = 0\,, \quad (\alpha : \beta) \in \mathbb{P}^1\,,$$

in projective space $\mathbb{P}^n(\mathbb{C})$, which is defined by $S_1 = 0$ in $\mathbb{P}^{n+1}(\mathbb{C})$. Here, S_i denotes the i-th elementary-symmetric polynomial in the space coordinates of $\mathbb{P}^{n+1}$:

$$S_i = \sum_{0 \leq j_1 < \ldots < j_i \leq n+1} x_{j_1} \cdot \ldots \cdot x_{j_i}, \qquad i = 1, \ldots, 5\,.$$

To determine the singular locus of each quintic in the pencil, it turns out to be convenient to rewrite $F_{(\alpha:\beta)}$ in terms of the *i-th power sums* in the coordinates x_j defined by

$$C_i := \sum_{j=0}^{n+1} x_j^i, \qquad i = 1, \ldots, 5.$$

Modulo S_1, we have the following identities:

$$\begin{aligned}
S_1 &= C_1,\\
S_2 &= -\tfrac{1}{2}C_2,\\
S_3 &= \tfrac{1}{3}C_3,\\
S_4 &= -\tfrac{1}{4}C_4 + \tfrac{1}{8}C_2^2,\\
S_5 &= \tfrac{1}{5}C_5 - \tfrac{1}{6}C_2C_3.
\end{aligned}$$

So the hyperquintic $Q = Q_{(\alpha:\beta)}$ is given by

$$F_{(\alpha:\beta)} = \alpha S_5 + \beta S_2 S_3 = \tfrac{\alpha}{5}C_5 - \tfrac{\alpha+\beta}{6}C_2C_3 = 0\,.$$

Since $F_{(0:1)} = -\frac{1}{6}C_2C_3 = S_2S_3$ clearly has the projective variety $S_2 = S_3 = 0$ as singular locus, we assume $\alpha \neq 0$. The singular points of the hyperquintics are those where the gradients of the defining equations in $\mathbb{P}^{n+1}$ are dependent. So we have

$$\begin{aligned}
\eta \text{ singular} \quad &\Leftrightarrow \quad \operatorname{rank}\begin{pmatrix} \partial_0 F_{(\alpha:\beta)}(\eta) & \dots & \partial_{n+1} F_{(\alpha:\beta)}(\eta) \\ \partial_0 S_1(\eta) & \dots & \partial_{n+1} S_1(\eta) \end{pmatrix} \leq 1 \\
&\Leftrightarrow \quad \operatorname{rank}\begin{pmatrix} \partial_0 F_{(\alpha:\beta)}(\eta) & \dots & \partial_{n+1} F_{(\alpha:\beta)}(\eta) \\ 1 & \dots & 1 \end{pmatrix} \leq 1 \\
&\Leftrightarrow \quad \exists\, \mu \in \mathbb{C} : \partial_i F_{(\alpha:\beta)}(\eta) = \mu\,, \qquad i = 0, \dots, n+1\,.
\end{aligned}$$

Hence, for all indices $i = 0, \dots, n+1$ we obtain

$$\sum_{j=0}^{n+1} \partial_j F_{(\alpha:\beta)}(\eta) = (n+2) \cdot \mu = (n+2) \cdot \partial_i F_{(\alpha:\beta)}(\eta),$$

which leads via $S_1 = 0$ to the following lemma.

Lemma 1 *Each coordinate η_i of a singularity η of the hyperquintic $Q_{(\alpha:\beta)}$ in $\mathbb{P}^n = V(S_1(x_0, \dots, x_{n+1}))$ is a root of*

$$P(X) := P_\lambda(X) := X^4 - X^2 \cdot \lambda C_2 - X \cdot \tfrac{2}{3}\lambda C_3 + \tfrac{1}{n+2}\big(\lambda C_2^2 - C_4\big) = 0,$$

where $\lambda := \frac{\alpha+\beta}{2\alpha}$.

□

Note that the sum of the four roots of $P(X)$ is zero since the term X^3 does not occur.

2 The Family of Σ_{10}-symmetric Hyperquintics in $\mathbb{P}^8$

We now specialize to the case $n = 8$. According to Lemma 1, each coordinate η_i of a singularity η of the hyperquintic $Q_{(\alpha:\beta)}$ in $\mathbb{P}^8$ satisfies

$$P(X) = X^4 - X^2 \cdot \lambda C_2 - X \cdot \tfrac{2}{3}\lambda C_3 + \tfrac{1}{10}\big(\lambda C_2^2 - C_4\big) = 0\,,$$

where $\lambda = \frac{\alpha+\beta}{2\alpha}$. A priori, there are 23 cases to check, since the 10 coordinates may be distributed over the four roots a, b, c, d of P as follows:

Case 1:	$10a$	Case 9:	$6a, 3b, c$	Case 17:	$4a, 4b, 2c$
Case 2:	$9a, b$	Case 10:	$6a, 2b, 2c$	Case 18:	$4a, 4b, c, d$
Case 3:	$8a, 2b$	Case 11:	$6a, 2b, c, d$	Case 19:	$4a, 3b, 3c$
Case 4:	$8a, b, c$	Case 12:	$5a, 5b$	Case 20:	$4a, 3b, 2c, d$
Case 5:	$7a, 3b$	Case 13:	$5a, 4b, c$	Case 21:	$4a, 2b, 2c, 2d$
Case 6:	$7a, 2b, c$	Case 14:	$5a, 3b, 2c$	Case 22:	$3a, 3b, 3c, d$
Case 7:	$7a, b, c, d$	Case 15:	$5a, 3b, c, d$	Case 23:	$3a, 3b, 2c, 2d$
Case 8:	$6a, 4b$	Case 16:	$5a, 2b, 2c, d$		

We analyse some example cases here; the remaining cases can be found in the appendix. First, we determine only the Σ_{10}-orbit length of the corresponding singularity η. Then, we further check for nodes in those cases that produced the longest orbits under Σ_{10}.

Case 1 does not occur, since on the one hand $\eta = (x : \ldots : x) \in \mathbb{P}^8$, and on the other hand the sum of its coordinates has to be zero.

Case 2 Assume that $\eta = (1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : -9)$. Hence $C_2 = 90$, $C_3 = -720$, $C_4 = 6570$, and

$$P(X) = X^4 - 90\lambda X^2 + 480\lambda X + 810\lambda - 657\,.$$

Requiring $P(1) = P(-9) = 0$, we obtain $\lambda = \frac{41}{75}$, thus

$$(\alpha : \beta) = (75 : 7)\,.$$

The length of the Σ_{10}-orbit of η is 10.

Case 7 A priori, we have $\eta = (a : a : a : a : a : a : a : b : c : d)$. Since $7a + b + c + d = 0$ and $a + b + c + d = 0$, we obtain $a = 0 = b + c + d$ and $\eta = (0 : 0 : 0 : 0 : 0 : 0 : 0 : b : c : -b - c)$. Since $b = c = 0$ is impossible, w.l.o.g. we put $b = 1$. By $P(0) = P(1) = P(c) = P(-1 - c) = 0$ we have $2\lambda = 1$, hence

$$(\alpha : \beta) = (1 : 0),$$

and no further conditions on $c \in \mathbb{C}$. Thus, we have found $120 = \frac{10 \cdot 9 \cdot 8}{3!}$ singular lines.

Case 12 We have $\eta = (1 : 1 : 1 : 1 : 1 : -1 : -1 : -1 : -1 : -1)$ and, thus, $C_2 = C_4 = 10$, $C_3 = 0$, and

$$P(X) = X^4 - 10\lambda X^2 + 10\lambda - 1 = \big(X^4 - 1\big) - 10\lambda\big(X^2 - 1\big)\,.$$

Hence, $P(\pm 1) = 0$ holds for all λ. This means that every single point in the Σ_{10}-orbit of η is a singularity of each hyperquintic $Q = Q_{(\alpha:\beta)}$ in the Σ_{10}-symmetric family in $\mathbb{P}^8$. For this reason, from now on we will call these points *generic singularities* (cf. [Schm]). The length of the Σ_{10}-orbit of η is $126 = \frac{1}{2} \cdot \binom{10}{5}$.

Case 18 Due to $4a + 4b + c + d = a + b + c + d = 0$ we immediately obtain $b = -a$ and $d = -c$, hence $\eta = (a : a : a : a : -a : -a : -a : -a : c : -c)$. Since $a = 0$ leads us back to case 4, we put $a = 1$ and find $C_2 = 2c^2 + 8$, $C_3 = 0$, $C_4 = 2c^4 + 8$, and P appropriate. Via $P(\pm 1) = P(\pm c) = 0$ we obtain

$$0 = \big(2\lambda(c^2 + 4) - (c^2 + 1)\big)(c + 1)(c - 1)\,.$$

With $c = \pm 1$ we are back in case 12, so we assume $c \neq \pm 1$. Thus,

$$0 = (2\lambda - 1)c^2 + (8\lambda - 1)\,.$$

This equation has no solution for $\lambda = \frac{1}{2}$, but for $\lambda \neq \frac{1}{2}$ we have

$$0 = \beta c^2 + (3\alpha + 4\beta)\,. \qquad (*)$$

Thus, $\eta = (1 : 1 : 1 : 1 : -1 : -1 : -1 : -1 : c : -c)$ is a singular point of $Q_{(\alpha:\beta)}$, $(\alpha : \beta) \neq (1 : 0)$, for all $c \in \mathbb{C}$ that satisfy $(*)$.

There are $3150 = \frac{1}{2} \cdot \binom{10}{4}\binom{6}{4}$ elements in the Σ_{10}-orbit of η, which we will also call *generic singularities* as well as in case 12 (cf. [Schm]).

If $(\alpha : \beta) \in \{\,(5 : -3), (4 : -3)\,\}$, which means $\lambda \in \{\,\frac{1}{5}, \frac{1}{8}\,\}$, the solutions of $(*)$ are $c = \pm 1$ and $c = 0$, respectively, so η coincides with the singular points from case 12 or two orbit elements of η merge to one of the singularities from case 17. Hence, we have singularities that are worse than ordinary nodes. A proof of this is given in section 3. For this reason, we from now on will refer to $(\alpha : \beta) \in \{\,(1 : 0), (5 : -3), (4 : -3)\,\}$ or $\lambda \in \{\,\frac{1}{2}, \frac{1}{5}, \frac{1}{8}\,\}$ as *exceptional values*. Case 7, however, already showed that $Q_{(1:0)}$ contains 120 singular lines.

We list the results of our investigation below. Table 1 shows the *generic singularities*, which are contained in each hyperquintic of the family. For the *exceptional values* $(\alpha : \beta) \in \{(1 : 0), (2 : -1), (3 : -2), (4 : -3), (5 : -3)\}$, the corresponding hyperquintics have singularities worse than ordinary nodes. In table 2 we list the parameter values, for which we have *additional* orbits of singular points. Using computer algebra we can verify that all the *additional* orbits consist only of ordinary nodes, if not stated otherwise.

orbit length	orbit element	case
126	$(1:1:1:1:1:-1:-1:-1:-1:-1)$	12
3150	$(1:1:1:1:-1:-1:-1:-1:c:-c)$, $\beta c^2 + (3\alpha + 4\beta) = 0$	18
12600	$(1:1:1:-1:-1:-1:c:c:-c:-c)$, $(\alpha + 2\beta)c^2 + (2\alpha + 3\beta) = 0$	23

Table 1: *Generic singularities* in $\mathbb{P}^8$. Each hyperquintic $Q_{(\alpha:\beta)}$ of the 1-parameter-family in $\mathbb{P}^8$ with $(\alpha : \beta)$ not an *exceptional value* contains these singular points.

As we will see in the next section, all the *generic singularities* are ordinary nodes. Moreover, for $(\alpha : \beta) = (3 : -1)$, which corresponds to the longest orbit of *additional* singular points, we find the best hyperquintic in the Σ_{10}-symmetric family in $\mathbb{P}^8$.

$(\alpha : \beta)$	orbit length	orbit element
$(3 : -1)$	7560	$(1:1:1:1:1:b_1:b_1:b_2:b_2:3)$, $b_{1,2} = -2 \pm \sqrt{-3}$
$(7 : -5)$	4200	$(3:3:3:3:b_1:b_1:b_1:b_2:b_2:b_2)$, $b_{1,2} = -2 \pm \sqrt{-3}$
$(51 : -25)$	2520	$(1:1:1:1:1:1:-5:-5:c_1:c_2)$, $c_{1,2} = 2 \pm \sqrt{-7/5}$
$(280 : -163 \pm 3\sqrt{65})$	2520	$(2:2:2:2:2:2b:2b:2b:-5-3b:-5-3b)$ $b = \frac{3\pm\sqrt{65}}{2}$
$(30 : -13 \mp \sqrt{85})$	1260	$(1:1:1:1:1:b:b:b:b:-4b-5)$, $b = \frac{-29\pm\sqrt{85}}{14}$
$(4 : -1)$	1260	$(1:1:1:1:1:1:b_1:b_1:b_2:b_2)$, $b_{1,2} = \frac{-3\pm\sqrt{-7}}{2}$
$(21 : 2b^3+25b^2+86b+76)$	840	$(1:1:1:1:1:1:b:b:b:-3b-6)$, $2b^4+25b^3+93b^2+139b+77=0$
$(84 : 175-3b(b^2+5b-13))$	360	$(1:1:1:1:1:1:1:b:b:-2b-7)$, $3b^4+39b^3+189b^2+413b+364=0$
$(50 : -37)$	210	$(2:2:2:2:2:2:-3:-3:-3:-3)$
$(175 : -117)$	120	$(3:3:3:3:3:3:3:-7:-7:-7)$
$(3 : 7)$	90	$(1:1:1:1:1:1:1:1:b_1:b_2)$, $b_{1,2} = -4 \pm \sqrt{-11}$
$(100 : -49)$	45	$(1:1:1:1:1:1:1:1:-4:-4)$
$(75 : 7)$	10	$(1:1:1:1:1:1:1:1:1:-9)$
$(1 : 0)$	120 lines	$(0:0:0:0:0:0:0:b:c:d)$, $b+c+d=0$
$(2 : -1)$	3150 lines	$(0:0:0:0:b:b:c:c:d:d)$, $b+c+d=0$
$(3 : -2)$	2800 lines	$(a:a:a:b:b:b:c:c:c:0)$, $a+b+c=0$
$(4 : -3)$	1575 (D_4) (Remark 2)	$(1:1:1:1:-1:-1:-1:-1:0:0)$
$(5 : -3)$	126 (Remark 1)	$(1:1:1:1:1:-1:-1:-1:-1:-1)$
$(0 : 1)$	hypersurface $S_2 = S_3 = 0$	

Table 2: Parameter values, for which we have *additional* orbits of singular points. By using computer algebra we can verify that only ordinary nodes are contained in these orbits, if not stated otherwise.

Theorem 1 *The hyperquintic $Q_{(3:-1)}$, given by*

$$3 \cdot S_5 + (-1) \cdot S_2 S_3 \;=\; S_1 \;=\; 0,$$

where $S_i,\ i = 1,2,3,5,$ is the i-th elementary-symmetric polynomial in 10 variables, has exactly 23436 ordinary nodes and no further singularities.

$\square$

3 Ordinary Nodes

To show that all the isolated singularities are ordinary nodes, we use the Hessian criterion, i.e. we show $\det(\mathrm{Hess}_f(y)) \neq 0$, where $\mathrm{Hess}_f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{ij}$ is the Hessian of f, $f = 0$ is the affine equation of the hyperquintic $Q_{(\alpha:\beta)}$ in an appropriate affine chart, and y is the singular point in this chart.

Modulo S_1 one has

$$\begin{aligned} F := F_{(1:\beta)} &= S_5 + \beta \cdot S_2 S_3 = \tfrac{1}{5} C_5 - \tfrac{1+\beta}{6} C_2 C_3 \\ &= \frac{1}{5}\left(\sum_{i=0}^{8} x_i^5 - g(x)^5\right) - \frac{1+\beta}{6}\left(\sum_{i=0}^{8} x_i^2 + g(x)^2\right)\left(\sum_{i=0}^{8} x_i^3 - g(x)^3\right), \end{aligned}$$

where $g(x) := x_0 + \ldots + x_8$. We consider the isolated singularities in affine charts $\mathbb{A}^8_i$, $i \in \{0, \ldots, 8\}$, given by

$$\begin{aligned} \mathbb{A}^8_i := \big\{ (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_8) | (x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_9) \in \mathbb{P}^8, \\ x_9 = -(x_0 + \ldots + x_8) \big\}. \end{aligned}$$

Those charts cover the projective space $\mathbb{P}^8$, so that we find all the isolated singularities in at least one chart $\mathbb{A}^8_i$. In our case it is even sufficient to check only one chart, w.l.o.g. $\mathbb{A}^8 := \mathbb{A}^8_0$, since no coordinate of our isolated singularities is zero. Defining $h(x) := 1 + x_1 + \ldots + x_8$, we obtain

$$\begin{aligned} f :=& f(x_1, \ldots, x_8) := \; F\big(1, x_1, \ldots, x_8, -(1 + x_1 + \ldots + x_8)\big) \\ =& \frac{1}{5}\left(1 + \sum_{k=1}^{8} x_k^5 - h(x)^5\right) - \frac{1+\beta}{6}\left(1 + \sum_{k=1}^{8} x_k^2 + h(x)^2\right)\left(1 + \sum_{k=1}^{8} x_k^3 - h(x)^3\right). \end{aligned}$$

Thus, it holds for the partial derivatives $f_i = \frac{\partial f}{\partial x_i}$, $i = 1, \ldots, 8$, of f

$$\begin{aligned} f_i = x_i^4 - h(x)^4 - \tfrac{1+\beta}{6}\Big[& 2(x_i + h(x)) + 3(x_i^2 - h(x)^2) + 2 \cdot \sum_{k=1}^{8} x_k^3 (x_i + h(x)) \\ &+ 3 \cdot \sum_{k=1}^{8} x_k^2 (x_i^2 - h(x)^2) - 2 x_i h(x)^3 + 3 x_i^2 h(x)^2 - 5 h(x)^4 \Big] \end{aligned}$$

and for the second partial derivatives $f_{ii} = \frac{\partial^2 f}{\partial x_i^2}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, $i \neq j$,

$$\begin{aligned} f_{ii} = 4x_i^3 - 4h(x)^3 - \tfrac{1+\beta}{6}\Big[&4 + 6x_i - 6h(x) + 12x_i^3 + 12x_i^2 h(x) + 4 \cdot \sum_{k=1}^{8} x_k^3 \\ &- 6x_i h(x)^2 + 6(x_i - h(x)) \sum_{k=1}^{8} x_k^2 - 22h(x)^3\Big], \\ f_{ij} = -4h(x)^3 - \tfrac{1+\beta}{6}\Big[&2 - 6h(x) + 6x_i x_j (x_i + x_j) + 6h(x)(x_i^2 + x_j^2) \\ &- 6h(x)^2 (x_i + x_j) + 2 \cdot \sum_{k=1}^{8} x_k^3 - 6h(x) \cdot \sum_{k=1}^{8} x_k^2 - 20h(x)^3\Big]. \end{aligned}$$

In the following subsections, we first check that all *generic singularities* are ordinary nodes. Then we verify that the longest orbit of length 7560 of the *additional* singularities of $Q_{(3:-1)}$ consists only of ordinary nodes.

3.1 The 126 generic Nodes

We consider $\eta := (1:1:1:1:1:-1:-1:-1:-1:-1)$ with its 126 orbit elements; due to our choice of the affine chart $\mathbb{A}^8$ and $S_1 = 0$, we evaluate the Hessian $\mathrm{Hess}_f = \big(f_{ij}\big)$ in $y := (1,1,1,1,-1,-1,-1,-1)$. With $h(y) = 1$, we obtain

$$f_{ii}(y) = \begin{cases} 0\,, & \text{if } i \leq 4, \\ 12 + 20\beta, & \text{if } i > 4, \end{cases} \qquad \text{and} \qquad f_{ij}(y) = 6 + 10\beta \text{ for all } i \neq j.$$

Thus,

$$\mathrm{Hess}_f(y) = \big(6 + 10\beta\big) \cdot \begin{pmatrix} 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & \ddots & & & & & \vdots \\ \vdots & \ddots & 0 & \ddots & & 1 & & \vdots \\ \vdots & & \ddots & 0 & \ddots & & & \vdots \\ \vdots & & & \ddots & 2 & \ddots & & \vdots \\ \vdots & & 1 & & \ddots & 2 & \ddots & \vdots \\ \vdots & & & & & \ddots & 2 & 1 \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 2 \end{pmatrix}.$$

The determinant of the righthand matrix is 1, hence

$$\det(\mathrm{Hess}_f(y)) \neq 0 \ \text{ for all } \beta \neq -\tfrac{3}{5}.$$

But $(\alpha : \beta) = (1 : -\frac{3}{5}) = (5 : -3)$ is one of the *exceptional values*, hence all of the 126 orbit elements of η are ordinary nodes of $Q_{(\alpha:\beta)}$, $(\alpha : \beta) \neq (5 : -3)$, $\alpha \neq 0$. For $(\alpha : \beta) = (5 : -3)$ we have singularities worse than ordinary nodes.

3.2 The 3150 generic Nodes

Now consider $\eta := (1 : 1 : 1 : 1 : -1 : -1 : -1 : c : -c : -1)$ and its orbit elements, where $\beta c^2 + (3 + 4\beta) = 0$, in the affine chart $\mathbb{A}^8$. We put $y := (1, 1, 1, -1, -1, -1, c, -c)$ and obtain

$$h(y) = (1 + y_1 + \ldots + y_8) = 1, \qquad \sum_{k=1}^{8} y_k^3 = 0, \qquad \sum_{k=1}^{8} y_k^2 = 6 + 2c^2.$$

Let

$$\begin{aligned}
b_1 &:= -4 + (1+\beta)(8 + 2c^2)\,, \\
b_4 &:= 6 + 10\beta\,, \\
b_5 &:= 4c^3 - 4 - (1+\beta)(4c^3 + 6c - 10)\,, \\
b_6 &:= -4c^3 - 4 + (1+\beta)(4c^3 + 6c + 10)\,,
\end{aligned}$$

then we have

$$\begin{array}{ll}
f_{11}(y) = f_{22}(y) = f_{33}(y) = 0, & f_{44}(y) = f_{55}(y) = f_{66}(y) = 2b_1, \\
f_{77}(y) = b_5, & f_{88}(y) = b_6,
\end{array}$$

and for $i \neq j$

$$f_{ij}(y) = \begin{cases}
b_1, & y_i = y_j = +1, \\
b_1, & y_i = y_j = -1, \\
b_1, & y_i = +1,\ y_j = -1, \\
b_4, & y_i = +1,\ y_j = +a, \\
b_4, & y_i = +1,\ y_j = -a, \\
b_1, & y_i = -1,\ y_j = +a, \\
b_1, & y_i = -1,\ y_j = -a, \\
b_4, & y_i = +a,\ y_j = -a.
\end{cases}$$

Hence, for the Hessian $\mathrm{Hess}_f(y)$ we have

$$\mathrm{Hess}_f(y) = \left(\begin{array}{ccc|ccc|c|c}
0 & b_1 & b_1 & b_1 & b_1 & b_1 & b_4 & b_4 \\
b_1 & 0 & b_1 & b_1 & b_1 & b_1 & b_4 & b_4 \\
b_1 & b_1 & 0 & b_1 & b_1 & b_1 & b_4 & b_4 \\
\hline
b_1 & b_1 & b_1 & 2b_1 & b_1 & b_1 & b_1 & b_1 \\
b_1 & b_1 & b_1 & b_1 & 2b_1 & b_1 & b_1 & b_1 \\
b_1 & b_1 & b_1 & b_1 & b_1 & 2b_1 & b_1 & b_1 \\
\hline
b_4 & b_4 & b_4 & b_1 & b_1 & b_1 & b_5 & b_4 \\
\hline
b_4 & b_4 & b_4 & b_1 & b_1 & b_1 & b_4 & b_6
\end{array}\right).$$

Performing row and column transformations, one easily finds

$$\det(\mathrm{Hess}_f(y)) = 2^8 \cdot c^2 \cdot (c^2 - 1)^8 \cdot \frac{3^2}{(c^2 + 4)^2}\,.$$

The denominator is not zero, since this would lead to a contradiction with the constraint on c. So the determinant only vanishes for $c \in \{0, \pm 1\}$. But c takes these values only for $(\alpha : \beta) \in \{(5 : -3), (4 : -3)\}$, which are *exceptional values*. Then we have singularities worse than ordinary nodes, due to certain merging singularities. For other values of $(\alpha : \beta)$, $\alpha \neq 0$, all the 3150 orbit elements of η are ordinary nodes.

3.3 The 12600 generic and the 7560 additional Nodes

For the 12600 orbit elements of $(1 : 1 : 1 : -1 : -1 : -1 : c : c : -c : -c)$ with $(1 + 2\beta) \cdot c^2 + (2 + 3\beta) = 0$ as well as for the 7560 *additional* orbit elements of $(1 : 1 : 1 : 1 : 1 : b_1 : b_1 : b_2 : b_2 : 3)$ with $b_{1,2} = -2 \pm \sqrt{-3}$, the procedure is exactly the same. For the latter case, we take $\beta = -\frac{1}{3}$ into account, since it is an *additional* orbit of singularities for $Q_{(3:-1)} = Q_{(1:-\frac{1}{3})}$. Thus, we find $\det(\mathrm{Hess}_f(y_1)) \neq 0$ for $(\alpha : \beta) \neq (5 : -3), (3 : -2)$, and $\det(\mathrm{Hess}_f(y_2)) \neq 0$, where $y_1 := (1, 1, -1, -1, c, c, -c, -c)$ and $y_2 := (1, 1, 1, 1, b_1, b_1, b_2, b_2)$. So the 12600 *generic* and the 7560 *additional* orbit elements of the corresponding singularities are ordinary nodes. The latter ones are contained only in $Q_{(3:-1)}$.

4 Concluding Remarks

We have proved in the previous sections that the hyperquintic $Q_{(3:-1)}$ in $\mathbb{P}^8$ has 23436 ordinary nodes and no further singularities. We now briefly discuss the cases $(\alpha : \beta)$, where $Q_{(\alpha:\beta)}$ has higher singularities (see table 2). Moreover, we look at the generalization of our approach to $\mathbb{P}^n$ and compare it to another construction of hyperquintics in $\mathbb{P}^n$ with many nodes.

4.1 Some Cases with higher Singularities

Remark 1 *For* $(\alpha : \beta) = (5 : -3)$*, 25 respectively 100 orbit elements of the* generic *singularities from cases 18 and 23, respectively, coincide with one appropriate orbit element of* $(1 : 1 : 1 : 1 : 1 : -1 : -1 : -1 : -1 : -1)$*. Thus, 126 singularities with Milnor number* $256 = 2^8$ *are created. The tangent cone of* $Q_{(5:-3)}$ *is a smooth cubic.*
We have similarities to this in the case of $\mathbb{P}^4$*; here,* $Q_{(3:-1)}$ *has exactly the 10 orbit elements of* $(1 : 1 : 1 : -1 : -1 : -1)$ *as singular points, they are called* Del Pezzo Nodes *in [vStr]. They have a Milnor number of* $16 = 2^4$*, the tangent cone of* $Q_{(3:-1)}$ *is a smooth cubic as well.*

Remark 2 *Besides the two orbits of* generic *ordinary nodes, the hyperquintic* $Q_{(4:-3)}$ *in* $\mathbb{P}^8$ *has one more orbit with 1575 isolated singularities of type* D_4*, namely the orbit elements of* $(1 : 1 : 1 : 1 : -1 : -1 : -1 : -1 : 0 : 0)$*. See also [Schm].*

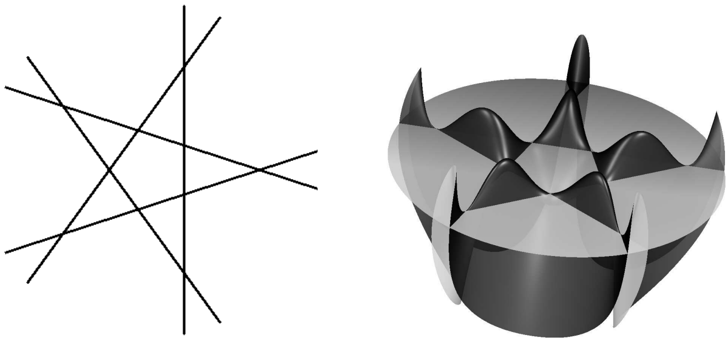

Figure 1: $R_5(x, y) = 0$ in $\mathbb{P}^2$ and $z - R_5(x, y) = 0$ in $\mathbb{P}^3$.

4.2 The Pentagon Construction

Theorem 1 improves the previously best known lower bound of 23126 for the maximum number of ordinary nodes a hyperquintic in $\mathbb{P}^8$ can have. The hypersurface corresponding to that previous lower bound was obtained with an approach based on a generalization of constructions by Givental for cubic hypersurfaces and Hirzebruch for quintics in $\mathbb{P}^4$ (cf. [AGZV], [Hir], and [Lab], sections 3.8–3.12). The basic idea is the usage of several polynomials of degree 5 in two variables, which have only a small number of critical values, to construct hyperquintics with many nodes. More precisely, one considers regular pentagons

$$R_5(x, y) := x^5 - 10x^3y^2 + 5xy^4 - 5x^4 - 10x^2y^2 - 5y^4 + 20x^2 + 20y^2 - 16$$

in the plane. These can be normalized such that their critical values are 0 and ± 1 (see figure 1). Then, Givental's equations for cubics can be transferred word by word to obtain an affine equation for hyperquintics in $\mathbb{P}^n$ with many singular points (all are nodes):

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^{j \cdot (1 + (n \bmod 2))} \widetilde{R}_5(x_{2j}, x_{2j+1}) = -(n \bmod 2) \frac{T_5(x_{n-1}) - 1}{2},$$

where $T_5(z) := 16z^5 - 20z^3 + 5z$ denotes the Tchebychev polynomial of degree 5 with two critical values ± 1 and $\widetilde{R}_5(x, y)$ is the normalized pentagon with critical value $+1$ over the origin. For a comparison of the resulting hyperquintics obtained by this method to our Σ_{n+2}-symmetric approach see table 3.

4.3 The Σ_{n+2}-symmetric Approach

We performed further experiments for some $n \neq 4$, and it seems to us that the Σ_{n+2}-symmetric construction yields fewer nodes than the pentagon construction. Indeed, the best hyperquintic $Q_{(7:-4)}$ in $\mathbb{P}^5$ contains only 210

n	number of ordinary nodes		$\mathrm{Ar}_n(5)$
	Σ_{n+2}-symmetric approach	pentagon construction	
3	20	31	31
4	**130**	126	135
5	210	420	456
6	1505	1620	1918
8	**23436**	23126	27876
10	296604	325580	411334

Table 3: Comparison of our Σ_{n+2}-symmetric approach and the pentagon construction in $\mathbb{P}^n$ for some n.

ordinary nodes (cf. table 3); in $\mathbb{P}^3$, the best hyperquintic $Q_{(2:1)}$ has only 20 ordinary nodes. For $n = 6$ and $n = 10$ we obtained 1505 respectively 296604 ordinary nodes for the best examples. In $\mathbb{P}^4$ and $\mathbb{P}^8$, the Σ_{n+2}-symmetric approach yields hyperquintics with a higher number of ordinary nodes than the pentagon construction (cf. table 3).
We did not look at other n in detail, we only verified that the number of *generic* nodes of the Σ_{n+2}-symmetric approach is less than the number of nodes obtained by using the pentagon construction. It is possible that for certain n and $(\alpha : \beta)$ the Σ_{n+2}-symmetric construction is better.

Appendix

The Case Analysis

Here we list the remaining cases of the case analysis in section 2.

Case 3 Assume that $\eta = (1:1:1:1:1:1:1:1:-4:-4)$. Hence $C_2 = 40$, $C_3 = -120$, $C_4 = 520$, and

$$P(X) = X^4 - 40\lambda X^2 + 80\lambda X + 160\lambda - 52\,.$$

Via $P(1) = P(-4) = 0$ we get $\lambda = \frac{51}{200}$, thus

$$(\alpha : \beta) = (100 : -49)\,.$$

The length of the Σ_{10}-orbit of η is $45 = \binom{10}{2}$.

Case 4 Consider $\eta = (a:a:a:a:a:a:a:a:b:-8a-b)$. For $a = 0$ we get $\eta = (0:0:0:0:0:0:0:0:1:-1)$, hence $C_2 = C_4 = 2$, $C_3 = 0$, and

$$P(X) = X^4 - 2\lambda X^2 + \tfrac{1}{5}(2\lambda - 1)\,.$$

Requiring $P(0) = P(\pm 1) = 0$ leads to $\lambda = \frac{1}{2}$, hence

$$(\alpha : \beta) = (1 : 0)\,.$$

The length of the Σ_{10}-orbit of η is $45 = \frac{1}{2} \cdot 10 \cdot 9$.

For $a \neq 0$ we have $\eta = (1:1:1:1:1:1:1:1:b:-8-b)$, hence $C_2 = 2b^2+16b+72$, $C_3 = -24b^2-192b-504$, $C_4 = 2b^4+32b^3+384b^2+2048b+4104$, and P appropriate. By requiring $P(1) = P(b) = P(-8-b) = 0$, we obtain three equations for λ:

$$\begin{aligned}
0 =& \lambda \cdot \big(\underbrace{2b^4 + 32b^3 + 342b^2 + 1712b + 3912}_{=:a_{11}}\big) \\
& + 1 \cdot \big(\underbrace{-b^4 - 16b^3 - 192b^2 - 1024b - 2047}_{=:a_{12}}\big) \\
0 =& \lambda \cdot \big(\underbrace{-8b^4 + 32b^3 + 552b^2 + 2832b + 2592}_{=:a_{21}}\big) \\
& + 1 \cdot \big(\underbrace{4b^4 - 16b^3 - 192b^2 - 1024b - 2052}_{=:a_{22}}\big) \\
0 =& \lambda \cdot \big(\underbrace{-8b^4 - 288b^3 - 3288b^2 - 16528b - 33888}_{=:a_{31}}\big) \\
& + 1 \cdot \big(\underbrace{4b^4 + 144b^3 + 1728b^2 + 9216b + 18428}_{=:a_{32}}\big)
\end{aligned}$$

To have a unique solution for λ, the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

must have rank 1. Thus, the three 2×2 minors

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

must vanish, which leads to

$$0 = (b+9)(b-1)(b+4)(b^2+8b+27)\,.$$

The solutions 1 and -9 lead us back to case 2, $b = -4$ to case 3. If we take one of the two roots $b = -4 \pm \sqrt{-11}$ of the remainig factor, $-8-b$ is the other one. Thus, the length of the Σ_{10}-orbit of η is $90 = 10 \cdot 9$.

Such an η yields $C_2 = 18$, $C_3 = 144$, $C_4 = -1350$, and

$$P(X) = X^4 - 18\lambda X^2 - 96\lambda X + \tfrac{162}{5}\lambda + 135\,.$$

Requiring $P(1) = P(-4 \pm i\sqrt{11}) = 0$ leads to $3\lambda = 5$ and

$$(\alpha : \beta) = (3 : 7)\,.$$

Case 5 Assume that $\eta = (3:3:3:3:3:3:3:-7:-7:-7)$. Hence $C_2 = 210$, $C_3 = -840$, $C_4 = 7770$, and

$$P(X) = X^4 - 210\lambda X^2 + 560\lambda X + 4410\lambda - 777\,.$$

From $P(3) = P(-7) = 0$ we get $\lambda = \frac{29}{175}$, hence

$$(\alpha:\beta) = (175:-117)\,.$$

The length of the Σ_{10}-orbit of η is $120 = \binom{10}{3}$.

Case 6 Consider $\eta = (a:a:a:a:a:a:a:b:b:-7a-2b)$. For $a=0$ we put $b=1$, hence $C_2 = 6 = -C_3$, $C_4 = 18$, and

$$P(X) = X^4 - 6\lambda X^2 + 4\lambda X + \tfrac{18}{5}\lambda - \tfrac{9}{5}\,.$$

$P(0) = P(1) = P(-2) = 0$ yields $\lambda = \frac{1}{2}$, hence

$$(\alpha:\beta) = (1:0)\,.$$

The length of the Σ_{10}-orbit of η is $360 = 10\cdot\binom{9}{2}$.

For $a \neq 0$, we have $\eta = (1:1:1:1:1:1:1:b:b:-7-2b)$. Thus,

$$\begin{aligned}
C_2 &= 6b^2 + 28b + 56,\\
C_3 &= -6b^3 - 84b^2 - 294b - 336,\\
C_4 &= 18b^4 + 224b^3 + 1176b^2 + 2744b + 2408,
\end{aligned}$$

and P appropriate. By requiring $P(1) = P(b) = P(-2b-7) = 0$, we again obtain three equations for λ (cf. case 4), hence a 3×2-matrix, which must have rank 1 to have a unique solution for λ. Thus, its three 2×2-minors must vanish and we find

$$0 = (b-1)(3b+7)(b+4)(3b^4 + 39b^3 + 189b^2 + 413b + 364)\,.$$

The solutions $b \in \{1, -4, -\frac{7}{3}\}$ lead us back into cases 2, 3, and 5, respectively. For b a root of the remaining factor and by $P(1) = P(b) = P(-2b-7) = 0$, we obtain $\lambda = \frac{-1}{168}(3b^3 + 15b^2 - 39b - 259)$, hence

$$(\alpha:\beta) = (84:-3b^3 - 15b^2 + 39b + 175)\,.$$

The length of the Σ_{10}-orbit of η is $360 = 10\cdot\binom{9}{2}$.

Case 8 We consider $\eta = (2:2:2:2:2:2:-3:-3:-3:-3)$ and obtain $C_2 = -C_3 = 60$, $C_4 = 420$, and, thus,

$$P(X) = X^4 - 60\lambda X^2 + 40\lambda X + 360\lambda - 42\,.$$

$P(2) = P(-3) = 0$ leads to $\lambda = \frac{13}{100}$, hence

$$(\alpha:\beta) = (50:-37)\,.$$

The length of the Σ_{10}-orbit of η is $210 = \binom{10}{4}$.

Case 9 Assume that $\eta = (a : a : a : a : a : a : b : b : b : -6a - 3b)$. For $a = 0$, $b = 1$ we find $C_2 = 12$, $C_3 = -24$, $C_4 = 84$, and

$$P(X) = X^4 - 12\lambda X^2 + 16\lambda X + \tfrac{72}{5}\lambda - \tfrac{42}{5}\,,$$

but for no λ does $P(0) = P(1) = P(-3) = 0$ hold simultaneously.

For $\eta = (1 : 1 : 1 : 1 : 1 : 1 : b : b : b : -3b - 6)$ we obtain

$$\begin{aligned} C_2 &= 12b^2 + 36b + 42, \\ C_3 &= -24b^3 - 162b^2 - 324b - 210, \\ C_4 &= 84b^4 + 648b^3 + 1944b^2 + 2592b + 1302 \end{aligned}$$

and P appropriate. Equating $P(X)$ to zero for $X = 1, b, -3b - 6$ leads to an equation system for λ again (cf. cases 4 and 6), hence a 3×2-matrix, whose three 2×2-minors must vanish to have a unique solution for λ. Thus,

$$0 = (3b + 7)(b - 1)(2b + 3)(b + 2)(2b^4 + 25b^3 + 93b^2 + 139b + 77)\,.$$

The first three factors take us back into cases 5, 2, and 8, respectively, for $b = -2$ we find $C_2 = -C_3 = 18$, $C_4 = 54$, and

$$P(X) = X^4 - 18\lambda X^2 + 12\lambda X + \tfrac{162}{5}\lambda - \tfrac{27}{5}\,.$$

Requiring $P(1) = P(-2) = P(0) = 0$ yields $\lambda = \frac{1}{6}$, hence

$$(\alpha : \beta) = (3 : -2)\,.$$

The length of the Σ_{10}-orbit of η is $840 = 10 \cdot \binom{9}{3}$.

For b a root of the remaining factor, we find $\lambda = \frac{1}{42}(2b^3 + 25b^2 + 86b + 97)$ by requiring $P(1) = P(b) = P(-3b - 6) = 0$. Thus,

$$(\alpha : \beta) = (21 : 2b^3 + 25b^2 + 86b + 76)\,.$$

The length of the Σ_{10}-orbit of η also is $840 = 10 \cdot \binom{9}{3}$ here.

Case 10 Assume that $\eta = (a : a : a : a : a : a : b : b : c : c)$ with $3a + b + c = 0$. For $a = 0$, $b = -c = 1$ we find $C_2 = C_4 = 4$, $C_3 = 0$, and

$$P(X) = X^4 - 4\lambda X^2 + \tfrac{8}{5}\lambda - \tfrac{2}{5}\,.$$

Via $P(0) = P(\pm 1) = 0$ we find $\lambda = \frac{1}{4}$, hence

$$(\alpha : \beta) = (2 : -1)\,.$$

The length of the Σ_{10}-orbit of η is $630 = \frac{1}{2} \cdot \binom{10}{2}\binom{8}{2}$.

For $a = 1$, $c = -3 - b$ we find $C_2 = 4(b^2 + 3b + 6)$, $C_3 = -6(3b^2 + 9b + 8)$, $C_4 = 4(b^4 + 6b^3 + 27b^2 + 54b + 42)$, and P appropriate. The conditions on the coordinates of η produce three equations for λ. By the same method as in cases 4, 6, and 9, we obtain

$$0 = (b + 4)(b - 1)(2b + 3)(b^2 + 3b + 4)\,.$$

The linear factors lead us back to cases 3 and 8. For a root $b = \frac{-3\pm\sqrt{-7}}{2}$ of the last factor, $-b-3$ is the other one. Hence, the length of the Σ_{10}-orbit of η is $1260 = \binom{10}{2}\binom{8}{2}$.

Such an $\eta = (2 : 2 : 2 : 2 : 2 : 2 : 2b : 2b : -6 - 2b : -6 - 2b)$ implies $C_2 = 32$, $C_3 = 192$, $C_4 = -896$, and

$$P(X) = X^4 - 32\lambda X^2 - 128\lambda X + \tfrac{512}{5}\lambda + \tfrac{448}{5}\,.$$

By $P(2) = P(2b) = P(-6 - 2b) = 0$ we find $\lambda = \frac{3}{8}$, hence

$$(\alpha : \beta) = (4 : -1)\,.$$

Case 11 Assume that $\eta = (a : a : a : a : a : a : b : b : c : d)$. Due to $6a + 2b + c + d = a + b + c + d = 0$, we obtain $b = -5a$ and $d = 4a - c$.

$a = 0$ takes us back to case 4, so we put $a = 1$. This leads to

$$\begin{aligned} C_2 &= 2c^2 - 8c + 72, \\ C_3 &= 12c^2 - 48c - 180, \\ C_4 &= 2c^4 - 16c^3 + 96c^2 - 256c + 1512, \end{aligned}$$

and P appropriate. By the conditions on P we find

$$5c^2 - 20c + 27 = 0 \quad\text{and}\quad 51\lambda = 13\,.$$

For a root $c = 2 \pm \sqrt{-\frac{7}{5}}$, the other one is $4 - c$. Hence, we obtain

$$(\alpha : \beta) = (51 : -25)$$

and an orbit length of η of $2520 = 10 \cdot 9 \cdot \binom{8}{2}$.

Case 13 Consider $\eta = (a : a : a : a : a : b : b : b : b : -5a - 4b)$. For $a = 0$ and $b = 1$ we find $C_2 = 20$, $C_3 = -60$, $C_4 = 260$, and

$$P(X) = X^4 - 20\lambda X^2 + 40\lambda X + 40\lambda - 26\,.$$

Requiring $P(0) = P(1) = P(-4) = 0$ leads to a contradiction.

For $a=1$ we have $C_2 = 20b^2+40b+30$, $C_3 = -60b^3-240b^2-300b-120$, $C_4 = 260b^4+1280b^3+2400b^2+2000b+630$, and P appropriate. If we require $P(1)=P(b)=P(-4b-5)=0$, we obtain the following equations:

$$\begin{aligned}0 &=(b-1)(b+1)^3(2b+3)(7b^2+29b+27)\,,\\ 0 &=(b+1)(450\lambda+182b^5+1055b^4+1795b^3+145b^2-2055b-1368)\,.\end{aligned}$$

The solutions $b\in\{1,-1,-\frac{3}{2}\}$ of the first equation lead us back to cases 2, 12, and 8, respectively, the roots $b=\frac{-29\pm\sqrt{85}}{14}$ of the remaining factor together with the second equation imply $\lambda=-\frac{7}{30}b-\frac{1}{5}=\frac{17\mp\sqrt{85}}{60}$, hence

$$(\alpha:\beta)=(30:-13\mp\sqrt{85})\,.$$

Here, the length of the Σ_{10}-orbit of η is $1260=10\cdot\binom{9}{4}$.

Case 14 Assume that $\eta=(a:a:a:a:a:b:b:b:c:c)$. For $a=0$, $b=2$ we obtain $C_2=-C_3=30$, $C_4=210$, and

$$P(X)=X^4-30\lambda X^2+20\lambda X+90\lambda-21\,.$$

Requiring $P(0)=P(2)=P(-3)=0$, however, leads to a contradiction.

For $\eta=(2:2:2:2:2:2b:2b:2b:-5-3b:-5-3b)$ we find $C_2 = 10(3b^2+6b+7)$, $C_3=-30(b+7)(b+1)^2$, $C_4=10(21b^4+108b^3+270b^2+300b+133)$, and P appropriate. Requiring $P(2)=P(2b)=P(-5-3b)=0$ implies

$$\begin{aligned}0 &= (b+1)(78400\lambda-186b^5+85b^4+4265b^3+6565b^2-6235b-24486)\,,\\ 0 &= (b-1)(b+1)^3(3b+7)(b^2-3b-14)\,.\end{aligned}$$

The roots $b\in\{\,1,-1,-\frac{7}{3}\,\}$ of the second equation take us back to cases 3, 12, and 5, respectively. The roots $b=\frac{3\pm\sqrt{65}}{2}$ of the remaining factor of the second equation, however, imply $\lambda=\frac{117\pm3\sqrt{65}}{560}$ via the first equation, hence

$$(\alpha:\beta)=(280:-163\pm3\sqrt{65})\,.$$

The length of the Σ_{10}-orbit of η is $2520=\binom{10}{2}\binom{8}{3}$.

Case 15 Due to $5a+3b+c+d=a+b+c+d=0$ we may assume $\eta=(a:a:a:a:a:-2a:-2a:-2a:c:a-c)$. For $a=0$ we are taken to case 4.

For $a=1$ we find $C_2=2(c^2-c+9)$, $C_3=3(c^2-c-6)$, $C_4=2(c^4-2c^3+3c^2-2c+27)$, and P appropriate. But $P(1)=P(-2)=P(c)=P(1-c)=0$ imply $0=c(c-1)$ and $6\lambda=1$, and both $c=0$ and $c=1$ lead to case 9.

Case 16 Due to $5a + 2b + 2c + d = a + b + c + d = 0$ we may assume $\eta = (a : a : a : a : a : b : b : -4a - b : -4a - b : 3a)$. Since $a = 0$ leads to case 10, we put $a = 1$ and obtain $C_2 = 2(b^2 + 8b + 23)$, $C_3 = -24(b+2)^2$, $C_4 = 4b^4 + 32b^3 + 192b^2 + 512b + 598$, and P appropriate. Requiring $P(1) = P(b) = P(3) = P(-4 - b) = 0$, we find $b^2 + 4b + 7 = 0$ and $\lambda = \frac{1}{3}$, hence

$$(\alpha : \beta) = (3 : -1) .$$

For $b = -2 \pm \sqrt{-3}$ one of the two solutions, $-4 - b$ is the other one, so we find $7560 = 10 \cdot \binom{9}{2}\binom{7}{2}$ elements in the Σ_{10}-orbit of η.

Case 17 Assume that $\eta = (a : a : a : a : b : b : b : b : -2a - 2b : -2a - 2b)$. For $a = 0$, $b = 1$ we find $C_2 = -C_3 = 12$, $C_4 = 36$, and

$$P(X) = X^4 - 12\lambda X^2 + 8\lambda X + \tfrac{72}{5}\lambda - \tfrac{18}{5} .$$

Via $P(0) = P(1) = P(-2) = 0$ we immediately obtain $\lambda = \frac{1}{4}$, hence

$$(\alpha : \beta) = (2 : -1) .$$

The length of the Σ_{10}-orbit of η is $3150 = \binom{10}{2}\binom{8}{4}$.

For $a = 1$ we find $C_2 = 4(3b^2 + 4b + 3)$, $C_3 = -12(b^3 + 4b^2 + 4b + 1)$, $C_4 = 4(9b^4 + 32b^3 + 48b^2 + 32b + 9)$, and P appropriate. Requiring $P(1) = P(b) = P(-2b - 2) = 0$, we obtain

$$\begin{aligned} 0 &= b(b-1)(b+1)^3(3b+2)(2b+3) , \\ 0 &= 400\lambda + 210b^6 + 695b^5 + 556b^4 - 377b^3 - 742b^2 - 344b - 100 . \qquad (*) \end{aligned}$$

The case $b = 0$ is checked above, whereas $b = 1$ and $b \in \{-\frac{2}{3}, -\frac{3}{2}\}$ take us back to cases 3 and 8, respectively. For $b = -1$ we obtain $\lambda = \frac{1}{8}$ via $(*)$, hence

$$(\alpha : \beta) = (4 : -3) .$$

The length of the Σ_{10}-orbit of η is $1575 = \frac{1}{2} \cdot \binom{10}{4}\binom{6}{4}$.

Case 19 We may assume $\eta = (3a : 3a : 3a : 3a : b : b : b : -4a - b : -4a - b : -4a - b)$. For $a = 0$, $b = 1$ we find $C_2 = C_4 = 6$, $C_3 = 0$, and

$$P(X) = X^4 - 6\lambda X^2 + \tfrac{18}{5}\lambda - \tfrac{3}{5} .$$

$P(0) = P(\pm 1) = 0$ leads to $\lambda = \frac{1}{6}$, hence

$$(\alpha : \beta) = (3 : -2) .$$

The length of the Σ_{10}-orbit of η is $2100 = \frac{1}{2} \cdot \binom{10}{3}\binom{7}{3}$.

For $a = 1$ we find $C_2 = 6(b^2 + 4b + 14)$, $C_3 = -12(3b^2 + 12b + 7)$, $C_4 = 6(b^4 + 8b^3 + 48b^2 + 128b + 182)$, and P appropriate. Requiring $P(3) = P(b) = P(-4-b) = 0$ leads to

$$0 = (b-3)(b+2)(b+7)(b^2+4b+7)\,,$$
$$0 = 4900\lambda - b^4 - 8b^3 - 6b^2 + 40b - 581\,.$$

The solutions $b \in \{-7, -2, 3\}$ of the first equation take us back to cases 5 and 8, respectively. If b is one of the two remaining roots $-2 \pm \sqrt{-3}$ of the first equation, $-4-b$ is the other one. Hence, the length of the Σ_{10}-orbit of η is $4200 = \binom{10}{3}\binom{7}{3}$. Furthermore, by the second equation we obtain $\lambda = \frac{1}{7}$, hence

$$(\alpha : \beta) = (7 : -5)\,.$$

Case 20 Due to $4a+3b+2c+d = a+b+c+d = 0$ we have $c = -3a-2b$, $d = 2a+b$, and $\eta = (a : a : a : a : b : b : b : -3a-2b : -3a-2b : 2a+b)$. Since $a = 0$ takes us to case 17, we put $a = 1$ and obtain

$$C_2 = 12b^2 + 28b + 26,$$
$$C_3 = -12b^3 - 66b^2 - 96b - 42,$$
$$C_4 = 36b^4 + 200b^3 + 456b^2 + 464b + 182,$$

and P appropriate. Equating $P(X)$ to zero for $X = 1, b, -3-2b, 2+b$, we find

$$0 = (b+1)^3(b+2)\,,$$
$$0 = (b+1)(300\lambda + 11b^3 + 18b^2 - 33b - 100)\,.$$

The two solutions $b \in \{-2, -1\}$ of the first equation lead us back to cases 9 and 12, respectively.

Case 21 Since we have $2a+b+c+d = a+b+c+d = 0$, we immediately may assume $\eta = (0 : 0 : 0 : 0 : b : b : c : c : -b-c : -b-c)$. As b and c cannot vanish simultaneously, w.l.o.g. we put $b = 1$ and find $C_2 = 4(c^2+c+1)$, $C_3 = -6c(c+1)$, $C_4 = 4(c^2+c+1)^2$, and P appropriate. $P(0) = P(1) = P(c) = P(-1-c) = 0$ imply $\lambda = \frac{1}{4}$, so

$$(\alpha : \beta) = (2 : -1)\,.$$

We thus have found $3150 = \frac{1}{3!}\binom{10}{2}\binom{8}{2}\binom{6}{2}$ singular lines, since there are no further conditions on $c \in \mathbb{C}$.

Case 22 Due to $3a+3b+3c+d = a+b+c+d = 0$ we have $a+b+c = 0 = d$ and, hence, $\eta = (a : a : a : b : b : b : -a-b : -a-b : -a-b : 0)$. Since $a = 0$ leads us back to case 19, we put $a = 1$ and find $C_2 = 6(b^2+b+1)$,

$C_3 = -9b(b+1)$, $C_4 = 6(b^2+b+1)^2$, and P appropriate. Requiring $P(0) = P(1) = P(b) = P(-1-b) = 0$, we obtain $\lambda = \frac{1}{6}$, hence

$$(\alpha : \beta) = (3 : -2)\,.$$

Since there are no further conditions on $b \in \mathbb{C}$, we again have found $2800 = \frac{1}{3!}\binom{10}{3}\binom{7}{3}\binom{4}{3}$ singular lines.

Case 23 Due to $3a+3b+2c+2d = a+b+c+d = 0$ we have $b = -a$, $d = -c$ and, hence, may assume $\eta = (a : a : a : -a : -a : -a : c : c : -c : -c)$. For $a = 0$ we are back in case 10, so we put $a = 1$. Thus, we find $C_2 = 4c^2+6$, $C_3 = 0$, $C_4 = 4c^4+6$, and P appropriate. Via $P(\pm c) = P(\pm 1) = 0$ we obtain

$$0 = \big(2\lambda(2c^2+3) - (c^2+1)\big)(c+1)(c-1)\,.$$

With $c = \pm 1$ we are back in case 12, so we assume $c \neq \pm 1$. Thus,

$$0 = (4\lambda - 1)c^2 + (6\lambda - 1)\,.$$

This equation has no solution for $\lambda = \frac{1}{4}$, but for $\lambda \neq \frac{1}{4}$ we have

$$0 = (\alpha + 2\beta)c^2 + (2\alpha + 3\beta)\,. \qquad (*)$$

So η is a singular point of $Q_{(\alpha:\beta)}$, $(\alpha : \beta) \neq (2 : -1)$, for all $c \in \mathbb{C}$ that satisfy $(*)$ and we have found $12600 = \frac{1}{2}\cdot\binom{10}{3}\binom{7}{3}\binom{4}{2}$ more *generic singularities* of $Q_{(\alpha:\beta)}$ in $\mathbb{P}^8$ (cf. [Schm] and the cases 12 and 18).

If $(\alpha : \beta) \in \{\,(5 : -3), (3 : -2)\,\}$, which means $\lambda \in \{\,\frac{1}{5}, \frac{1}{6}\,\}$, the solutions of $(*)$ are $c = \pm 1$ and $c = 0$, respectively, so two respective orbit elements of η merge or they coincide with the singular points from case 12. Hence, we have singularities that are worse than ordinary nodes. A proof of this is given in section 3.
For this reason, we add $(\alpha : \beta) \in \{\,(2 : -1), (5 : -3), (3 : -2)\,\}$ and $\lambda \in \{\,\frac{1}{4}, \frac{1}{5}, \frac{1}{6}\,\}$ to the *exceptional values* introduced in case 18. The cases 21 and 22, however, already showed that we have singular lines contained in $Q_{(2:-1)}$ and $Q_{(3:-2)}$.

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