DEFORMATIONS OF ELLIPTIC FIBRE BUNDLES IN POSITIVE CHARACTERISTIC

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Abstract. We study deformation theory of elliptic fibre bundles over curves in positive characteristics. As an application, we give examples of non-liftable elliptic surfaces in charactertic two and three, which answers a question of Katsura and Ueno. Also, we classify deformations of bielliptic surfaces.

1. Introduction

In their seminal paper on elliptic surfaces in characteristic p [KU85], Katsura and Ueno asked if every elliptic surface of Kodaira dimension one, over a field of positive characteristic, can be lifted to characteristic zero.

Usually, one proves non-liftabilty for a given surface X by showing that certain numerical invariants of X, which are preserved under deformations, cannot be achieved in characteristic zero. For example, if X is a surface of general type over $\mathbb{C}$, one has the Bogomolov-Miyaoka-Yau inequality, which implies:

$$K_X^2/\chi(\mathscr{O}_X) \leq 9$$

Examples of surfaces, violating Bogomolov-Miyaoka-Yau, have been constructed by Szpiro, Hirzebruch and others (see [Lie09, Section 7] for an overview).

What is the situation in Kodaira dimension one? Recall that every surface of this class has a unique elliptic or quasi-elliptic fibration. A fibration is called quasi-elliptic if the generic fibre is a cuspidal curve of arithmetic genus one.

In comparison to the theory of surfaces of general type, numerical invariants do not play a central role in Kodaria dimension one. Namely, one always has $K_X^2 = 0$ and if X is elliptic and not quasi-elliptic than $\chi(\mathscr{O}_X) \geq 0$ holds. The last fact was used by Raynaud: In [Ray78] he constructed quasi-elliptic surfaces with $\chi(\mathscr{O}_X) < 0$, which are therefore non-liftable.

Our approach to construct non-liftable *elliptic* surfaces is entirely different. Instead of using invariants, we classify all possible deformations of a given surface, and a posteriori conclude that there are only deformations over rings in which p is zero. In particular, there is no lifting to characterstic zero.

To make this work, we have to chose a class of surfaces, with a sufficiently easy deformation theory, but being on the other hand rich enough, to provide the examples we are looking for. As it turns out, the right objects are elliptic fibre bundles over curves.

Let us fix a field k, and a curve C over k. By an elliptic fibre bundle over C, we understand an elliptic fibration $X \to C$ which is locally trivial for the étale topology. That is, every point $x \in C$ has an étale neighborhood $U \to C$ such that $X \times_C U$ is isomorphic to a product $U \times_k E$, where E is an elliptic curve over k.

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An elliptic fibre bundle is called *Jacobian* if it has a section. Examples of Jacobian elliptic bundles can be constructed as follows: Let E be an elliptic curve over k, and let Γ be a finite group acting on the group scheme E. Given a curve C/k with a free Γ action, we can form

$$X = (E \times C)/\Gamma, \tag{1.1}$$

where the action on the product is diagonal. The quotient X has a smooth Jacobian elliptic fibration $X \to C/\Gamma$.

We will classify deformations of elliptic fibre bundles over curves in several steps. First, we study Jacobian bundles, and prove that their deformations are always of the form (1.1).

Next, we study the relation between Jacobian and the non-Jacobian bundles. For an arbitrary elliptic bundle $f\colon X \to C$ denote by $\mathcal{F}ib_{X/C}$ the functor of deformations of X extending its fibration structure. Associated to X is a Jacobian bundle over the same base, which we denote by J. Let $\mathcal{J}ac_{J/C}$ be the subfunctor of $\mathcal{F}ib_{J/C}$ of deformations that have a section. For precise definitions of these functors see Definition 4.1.

There is a natural map $\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}$ given by taking the zero component of the relative Picard scheme.

Theorem (4.3)**.** *The map of deformation functors $\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}$ is formally smooth and we have an equation of vectorspace dimensions*

$$\dim(\mathcal{F}ib_{X/C}(k[\epsilon])) = \dim(\mathcal{J}ac_{X/C}(k[\epsilon])) + h^1(C, \mathrm{Lie}(J/C)).$$

This resembles situation of elliptic fibration over a field: The Jacobian fibrations can be dealt with explicitly and the non Jacobian ones are described by a cohomological theory, based on the group structure of the former.

Next, we answer the question whether, given an elliptic bundle $X \to C$, every deformation admits an extension of the fibration structure:

Theorem (5.1)**.** *If X is of Kodaira dimension one, then the unique elliptic fibration extends to every deformation.*

Now we can address the liftability question: In section 3.1 we will construct Jacobian bundles over fields of characteristic two and three, which cannot be lifted as Jacobian bundles. By the above theorems, this is enough to show non-liftabilty.

Theorem (3.7)**.** *There exist elliptic fibre bundles in characteristic two and three, that do not lift to characteristic zero.*

As a further application of the theory, we treat the case of bielliptic surfaces. This case is in a certain sense more interesting than the Kodaira dimension one case because one has to use methods from the deformation theory of abelian schemes. Our main result is:

Theorem (6.9)**.** *If X is a bielliptic surface, then both elliptic fibration extend under deformations. In other words: Every deformation of a bielliptic surface is bielliptic.*

For small p one encounters phenomena, which do not appear when considering the same class of surfaces in characteristic zero. For example, deformations become obstructed which was already observed by W. Lang in [Lan95]. There is also the possibility of deforming a Jacobian bielliptic surface into a non-Jacobian one, which is absent in characteristic zero.

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2. Preliminaries

In this section we introduce some standard techniques which will be used later on. Concerning deformation theory, we follow Schlesinger's fundamental paper [Sch68] in terminology, and freely make use of basic facts about pro-representable hulls of deformation functors.

A key problem that will appear in Sections 5 and 6 is of the following form: Given a deformation $\mathcal{X}$ of some scheme X, what properties and additional structures carry over to $\mathcal{X}$?

Let S be a scheme. For an S-scheme X, we consider the category $\mathcal{E}t_S(X)$ of S-schemes with a finite and étale S-map to X ("rěvetement étale"). It turns out that this category is invariant under nilpotend thickenings:

2.1. **Theorem** (SGA1 Theorem 5.5 and Theorem 8.3 [SGA63])**.** *Let $\mathcal{S}$ be a scheme with a closed subscheme S_0 having the same topological space as $\mathcal{S}$ itself. Let $\mathcal{X}$ be an $\mathcal{S}$-scheme, and denote by $X_0 = \mathcal{X} \times_{\mathcal{S}} S_0$ the restriction to S_0. Then the functor $\mathcal{E}t_{\mathcal{S}}(\mathcal{X}) \to \mathcal{E}t_{S_0}(X_0)$ given by*

$$\mathcal{Y} \mapsto \mathcal{Y} \times_{\mathcal{S}} S_0$$

is an equivalence of categories.

This theorem can be seen as a geometric form of Hensel's lemma from commutative algebra. We note two special cases: The categories of étale Galois covering of $\mathcal{X}$ and X_0 are equivalent, and so are the categories of finite étale group schemes. Recall that $S' \to S$ is called Galois with group Γ if Γ acts on S' as an S-scheme and we have an isomorphism

$$\Gamma \times S' \simeq S' \times_S S' \text{ given by } (\sigma, x) \mapsto (\sigma(x), x).$$

2.0.1. *Notations.* Finally, let us fix some notations. By k we denote an algebraically closed field of characteristic $p > 0$ if not stated otherwise. We denote by $W = W(k)$ its ring of Witt vectors. Let $\mathcal{A}lg$ be the category of local artin W-algebras having residue field k. Every scheme will be assumed notherian, and by a curve over some base scheme S we will always mean a proper, smooth and connected S-scheme.

3. Deformations of Jacobian bundles

In this Section, let R be a complete notherian local ring with residue field k. We denote by $\mathcal{S}$ a flat, integral and projective R-scheme. Let $\mathcal{J}$ be an elliptic scheme over $\mathcal{S}$, and let S_0 be the "reduction" $\mathcal{S} \times_R k$ of $\mathcal{S}$ and likewise set $J_0 = \mathcal{J} \times_{\mathcal{S}} S_0$. Our treatment of Jacobian bundles is based on the following statement, which is a simple application of the moduli theory of elliptic curves.

3.1. **Proposition.** *For some integer $N \geq 3$ which is prime to p, assume that the N-torsion subgroup scheme of $\mathcal{J}$ is split i.e; there is an isomorphism $\mathcal{J}[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$. Then there exists an elliptic curve $\mathcal{E}$ over R such that $\mathcal{J}$ is isomorphic to $\mathcal{E} \times_R \mathcal{S}$.*

Proof. Let us fix a level-N structure; i.e. an isomorphism $\mathcal{J}[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$. From [KM85, Corollary 4.7.2] we know that there exists a fine moduli space $\mathcal{M}$ of elliptic schemes with a fixed level-N structure.

This means that we get a morphisms $c\colon \mathcal{S} \to \mathcal{M}$ over R such that $\mathcal{J} \simeq c^*(\mathcal{E}^{univ})$ where $\mathcal{E}^{univ}$ is the universal family of the moduli problem; i.e. $\mathcal{E}^{univ}$ is an elliptic scheme with a level-N structure, but the level structure will no longer be relevant for us.

Again by [KM85, Corollary 4.7.2] we know that $\mathcal{M}$ is affine. The morphism c is therefore given by an R-algebra homomorphism

$$H^0(\mathcal{M}, \mathscr{O}_{\mathcal{M}}) \to H^0(\mathcal{S}, \mathscr{O}_{\mathcal{S}}) = R.$$

In other words, c factors over $\mathrm{Spec}(R)$ and therefore $\mathcal{J}$ is just the pullback of an elliptic curve $\mathcal{E}$ over R. □

We use this to classify deformations of Jacobian fibre bundles:

3.2. **Proposition.** *Let $\mathcal{S}' \to \mathcal{S}$ be a finite and étale Galois covering with group Γ such that $\mathcal{J}[N] \times_{\mathcal{S}} \mathcal{S}' \simeq (\mathbb{Z}/N\mathbb{Z})^2$. Then*

$$\mathcal{J} \simeq (\mathcal{E} \times_R \mathcal{S}')/\Gamma,$$

where $\mathcal{E}$ is an elliptic over R, and the action is the diagonal action given by the Galois action on $\mathcal{S}'$ and by a homomorphism $\Gamma \to \mathrm{Aut}(\mathcal{E})$ on the left factor.

Proof. The fibre bundle $\mathcal{J} \times_{\mathcal{S}} \mathcal{S}'$ satisfies the assumptions of Proposition 3.1. Thus there exists an elliptic curve $\mathcal{E}$ over R and an isomorphism $\mathcal{J} \times_{\mathcal{S}} \mathcal{S}' \simeq \mathcal{E} \times_R S'$.

In other words, the two fibrations $\mathcal{J}$ and $\mathcal{E} \times_R \mathcal{S}$ are twists of each other, becoming isomorphic after base change with $\mathcal{S}' \to \mathcal{S}$. Twists of the fibration $\mathcal{E} \times_R \mathcal{S}$ are classified up to isomorphism by the Galois cohomology set $H^1(\Gamma, A(\mathcal{S}'))$, where A is the group scheme $\mathrm{Aut}(\mathcal{E} \times_R \mathcal{S}')$ and we consider its $\mathcal{S}'$-valued points as Galois module under Γ.

We claim that the Galois action on A is trivial: For a suitable integer $N \geq 3$ and prime to p we have a closed immersion $A \subset \mathrm{Aut}(\mathcal{E}[N] \times_R \mathcal{S}')$. However, since R is a strict henslian ring, we find that $\mathcal{E}[N] \times_R \mathcal{S}'$ is the constant group scheme $(\mathbb{Z}/N\mathbb{Z})^2$ on $\mathcal{S}'$ and futhermore we see that $\mathrm{Aut}(\mathcal{E}[N] \times_R \mathcal{S}')$ is the constant group scheme $\mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathcal{S}'$.

Finite étale group schemes over $\mathcal{S}$ correspond to finite abstract groups with a continuous $\pi_1(\mathcal{S})$-action. We saw that A can be embedded into a group scheme with trivial $\pi_1(\mathcal{S})$-action, hence the action on A has to be trivial as well. The action of Γ on A is an induced action of a finite quotient $\pi_1(\mathcal{S}) \twoheadrightarrow \Gamma$, and therefore trival as well. Thus we have

$$H^1(\Gamma, \mathrm{Aut}(\mathcal{E} \times_R \mathcal{S}')(\mathcal{S}')) \simeq \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mathcal{E} \times_R \mathcal{S}')(\mathcal{S}')).$$

For a homomorphism ρ in the above group, the corresponding twist looks like $(\mathcal{E} \times_R \mathcal{S}')/\Gamma$, where the action of $\sigma \in \Gamma$ is given by

$$(x, y) \mapsto (\rho(\sigma)(x), \sigma y).$$

□

We can use these results to give a necessary and sufficient criterion for the existence of Jacobian liftings:

3.3. **Corollary.** *Let J be a Jacobian fibration over S, given by $(E \times_k S')/\Gamma$ for some étale Galois covering $S' \to S$ with group Γ. Denote the action of Γ on E by ρ_0. Then there exists a lifting $\mathcal{J} \to \mathcal{S}$ if and only if there exists a lifting $\mathcal{E}$ of E over R together with an extension of the action ρ_0.*

Proof. To show sufficiency is easy. The covering $S' \to S$ lifts uniquely to $\mathcal{S}' \to \mathcal{S}$ which is again Galois with group Γ. If a lifting $\mathcal{E}$ of E with the prescribed properties exists, simply put $\mathcal{J} = (\mathcal{E} \times_R \mathcal{S}')/\Gamma$. This quotient will exist in the category of schemes because Γ is finite.

In order to show necessity, assume that we have a lifting $\mathcal{J} \to \mathcal{S}$. Like before, we also have the unique lifting $\mathcal{S}' \to \mathcal{S}$ of the Galois covering. Observe that $\mathcal{J}[N] \times_{\mathcal{S}} \mathcal{S}'$ is split, since $\mathcal{J}[N]$ is a finite and étale group scheme and the reduction is split by assumption. Using Proposition 3.2, we find that $\mathcal{J} \simeq (\mathcal{E} \times_R \mathcal{S}')/\Gamma$, where the action of Γ on $\mathcal{E}$ is denoted by ρ. We claim that ρ lifts the action ρ_0:

Consider the induced action of ρ on $\mathcal{E}[N]$ for some integer N. The categories of étale group schemes over k and R are equivalent, hence ρ is determined by its action on the reduction $E[N]$.

For $N \geq 3$ we know that the group homomorphisms, given by restricting the automorphism group of an elliptic scheme to its N-torsion is injective [KM85, Corollary 2.7.2]. However the isomorphism type of $J[N]$ allows to read of the action of Γ on $J[N]$, for it is given by a class in

$$H^1(\Gamma, \mathrm{Aut}(J[N])(S')) \simeq \mathrm{Hom}(\Gamma, \mathrm{Aut}(E[N])(S'))$$

and the element of the latter group which corresponds to $J[N]$ is just ρ_0. Hence the restriction of ρ to the reduction has to be ρ_0. □

3.1. **Non-liftable elliptic surfaces.** We postpone the development of the general theory at this point to give some specific examples of Jacobian elliptic bundles that do not have a lifting to characteristic zero.

3.1.1. *Characterstic three.* For the first example, let k be an algebraically closed field of characteristic three, and let E be an elliptic curve over k with j-invariant 0. By [Sil09, Appendix A, Proposition 1.2] the automorphism group G of E is a semidirect product $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ where $\mathbb{Z}/4\mathbb{Z}$ acts on $\mathbb{Z}/3\mathbb{Z}$ in the unique non trivial way.

As we shall see later on, there exists a smooth and proper curve C over k such there is a surjection $\pi_1(C) \twoheadrightarrow G$. Denote by $C' \to C$ the associated finite and étale Galois cover. Now we set

$$J = (E \times_k C')/G,$$

where the action of G on E is the action of the automorphism group.

3.4. **Lemma.** *Let the characterstic of k be three, and let Λ be in $\mathcal{A}lg$ (see 2.0.1). If for an elliptic curve $\mathcal{E}$ over Λ the order of the automorphism group of $\mathcal{E}$ is greater than six, it follows $3 \cdot \Lambda = 0$.*

Proof. Assume by contradiction, that the order of $\mathrm{Aut}_0(\mathcal{E})$ is greater than six. Since two is a unit, there is a Weierstraß equation for $\mathcal{E}$ of the following form:

$$y^2 = x^3 + a_2x^2 + a_4x + a_6$$

Admissible transformations look like $x \mapsto u^2x + r$ and $y \mapsto u^3y + u^2sx + t$. The specific form of the equation implies $t = 0$ and $s = 0$. Standart arguments show that either $u^4 = 1$ or $u^6 = 1$. Thus, an automorphism group of order greater than six would have to contain an element of the form $x \mapsto x + r$.

We get an equation $a_2 = a_2 + 3r$, which implies $3r = 0$. But r has to be a unit, for otherwise the reduction map would not be injective on the automorphism group. Thus $3 = 0$ follows. □

Now we get as a direct consequence of Corollary 3.3:

3.5. **Proposition.** *The elliptic bundle J can be lifted (as Jacobian fibration) only over rings in which $3 = 0$ holds.*

3.1.2. *Characteristic two.* Now assume k is a characteristic two. Given an elliptic curve E over k with j-invariant 0, the group of automorphisms will be a semidirect product $G = Q \rtimes \mathbb{Z}/3\mathbb{Z}$, where Q is the quaternion group. Similarly to Lemma 3.4, one shows that neither G nor Q can lift to rings with $2 \neq 0$. Now assume the existence of two curves C_G and C_Q together with étale Galois covers $C'_G \to C_G$ of group G and $C'_Q \to C_Q$ of group Q respectively.

This gives rise to two Jacobian bundles: $J_G \simeq (C'_G \times E)/G$ and $J_Q \simeq (C'_Q \times E)/Q$. Again by Corollary 3.3 it follows:

3.6. **Proposition.** *The elliptic bundles J_G and J_Q can be lifted (as Jacobian fibrations) only over rings in which $2 = 0$ holds.*

Now we can state the main theorem for all the Jacobian bundles constructed in this section:

3.7. **Theorem.** *The bundles J (in characteristic three) and J_G, J_Q (in characteristic two) do not admit a formal lifting to characteristic zero.*

Proof. We already saw that J_G and J_Q cannot be lifted as Jacobian elliptic fibre bundles. From Proposition 4.1 below, it follows that the same is true for liftings which are not Jacobian but admit an elliptic fibration. Finally, observe that the base curves on both cases are of genus $g \geq 2$, which implies via canonical bundle formular, that the Kodaira dimension of J_Q and J_Q are 1. Now, by Theorem 5.1 below, we get that every deformation is elliptic. □

To finish this discussion, we have to establish the existence of curves with specific étale Galois coverings.

To this end, we use a powerful theory which is developed in [PS00]. First we fix some group theoretic invariants. Let G be a group with the property that the maximal p-Sylow subgroup P is normal. We set $H = G/P$. Then one can write G as a semidirect product $P \rtimes H$.

We denote by $\mathcal{P}$ the maximal elementary abelian quotient of P, and consider it as a $\mathbb{F}_p$-vector space, for it is a p-torsion group. Let $Z(H)$ be the set of irreducible characters with values in k, and let V_χ be an irreducible k-representation of H with character χ. On P, we have an H action coming from the structure of the semidirect product. This induces an H-representation on $\mathcal{P}$. Write

$$\mathcal{P} \otimes_{\mathbb{F}_p} k \simeq \oplus V_\chi^{m_\chi}.$$

The m_χ are thus numerical invariants of the group G.

Theorem (Theorem 7.4 [PS00])**.** *Let G be a group having a normal p-Sylow subgroup P. Suppose $H = G/P$ is abelian. Then there exists a curve of genus $g \geq 2$ having an étale Galois covering with group G if the minimal number of generators of H is less or equal than $2g$, and $m_\chi \leq g - 1$ holds for every $\chi \in Z(H)$.*

In the characteristic three example we had $G = \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. The minimal number of generator of $H = \mathbb{Z}/4\mathbb{Z}$ is one, the action of H on P is obviously irreducible and given by the sign involution. Thus the assumption of the theorem are satisfied for some curve of genus 2.

In the characteristic two examples we also have that the maximal p-Sylow group is normal. Thus for g sufficiently large, we will find curves with the required coverings.

4. Deformations of elliptic torsors

We start with some general theory on deformations of torsors under smooth commutative group schemes. This paragraph mainly rephrases [SGA66, Remarque 9.1.9]. We work in the following setting: Fix a small extension of algebras in $\mathcal{A}lg$

$$0 \to I \to \Lambda \to \Lambda_0 \to 0.$$

Let $\mathcal{S}$ be a flat Λ-scheme, $\mathcal{S}_0 = \mathcal{S} \otimes_\Lambda \Lambda_0$ its reduction over Λ_0. We have a closed immersion $i \colon \mathcal{S}_0 \to \mathcal{S}$. Let $\mathcal{G}$ be a smooth commutative $\mathcal{S}$-group scheme and $\mathcal{G}_0 = \mathcal{G} \times_{\mathcal{S}} \mathcal{S}_0$ the reduction over Λ_0.

For a group functor $\mathcal{F}$ on the category of $\mathcal{S}_0$ schemes, we defined the pushforward functor $i_*\mathcal{F}$ on $\mathcal{S}$-scheme by sending a $\mathcal{S}$-scheme $\mathcal{T}$ to $\mathcal{F}(\mathcal{T} \times_{\mathcal{S}} \mathcal{S}_0)$.

There is a natural specialization map $s \colon \mathcal{G} \to i_*\mathcal{G}_0$ of group functors. To investigate its kernel, we introduce a coherent sheaf on $\mathcal{S}_0$:

$$\mathscr{L} = \mathrm{Lie}(\mathcal{G}_0/\mathcal{S}_0) \otimes_{\mathscr{O}_{\mathcal{S}_0}} I.$$

We have a sequence of group functors on $\mathcal{S}$ namely

$$0 \to i_*\mathscr{L} \to \mathcal{G} \xrightarrow{s} i_*\mathcal{G}_0 \to 0, \tag{4.1}$$

whose exactness follows from the smoothness of $\mathcal{G}_0$, as can be seen affine locally. Taking étale cohomology of (4.1), we obtain the fundamental long exact sequence

$$0 \to i_*\mathcal{G}_0(\mathcal{S})/s(\mathcal{G}(\mathcal{S})) \to H^1(\mathcal{S}, i_*\mathscr{L}) \to H^1(\mathcal{S},\mathcal{G}) \xrightarrow{s} \to H^1(\mathcal{S}, i_*\mathcal{G}_0) \to H^2(\mathcal{S}, i_*\mathscr{L}). \tag{4.2}$$

The sheaves $\mathcal{L}$ and $i_*\mathcal{L}$ are coherent modules. We find

$$H^i(\mathcal{S}, i_*\mathscr{L}) \simeq H^i(\mathcal{S}_0, \mathscr{L}) \simeq H^i_{zar}(\mathcal{S}, \mathrm{Lie}(G_0/S_0)) \otimes I.$$

Furthermore, we claim that the group $H^1(\mathcal{S}, i_*\mathcal{G}_0)$ is isomorphic to $H^1(\mathcal{S},\mathcal{G}_0)$: By the Leray spectral sequence we get an exact sequence of étale cohomology groups

$$0 \to H^1(\mathcal{S}, i_*\mathcal{G}_0) \to H^1(\mathcal{S}_0, \mathcal{G}_0) \to H^0(\mathcal{S}_0, R^1 i_*\mathcal{G}_0).$$

We claim, that the last term vanishes: It is enough to show that $(R^1 i_*\mathcal{G}_0)_x = 0$ for every closed point x of $\mathcal{S}$; i.e. of $\mathcal{S}_0$. By [Mil80, Theorem 1.15] it follows that $(R^1 i_*\mathcal{G}_0)_x \simeq H^1(\mathrm{Spec}(\mathscr{O}^\wedge_{\mathcal{S}_0,x}), \mathcal{G}_0)$ and the last group vanishes since $\mathrm{Spec}(\mathscr{O}^\wedge_{\mathcal{S}_0,x})$ has only the trivial étale covering. Recall, that k is algebraically closed.

In our situation, this means the following: Let $J \to C$ be an Jacobian elliptic bundle over a curve C over k. Let $\mathcal{J}_0 \to \mathcal{C}_0$ be a Jacobian lifting over $\Lambda_0 \in \mathcal{A}lg$. As above, we fix a small extension

$$0 \to I \to \Lambda \to \Lambda_0 \to 0.$$

4.1. **Proposition.** *Let $\mathcal{J} \to \mathcal{C}$ be an Jacobian lifting of $\mathcal{J}_0/\mathcal{C}_0$ to Λ. Then every $\mathcal{J}_0$-torsor $\mathcal{X}_0$ over $\mathcal{C}_0$ lifts to a $\mathcal{J}$-torsor over $\mathcal{C}$. Futhermore, let m be an integer prime to p. Then we have an isomorphism of m-torsion groups:*

$$H^1(\mathcal{C}, \mathcal{J})[m] \xrightarrow{\sim} H^1(\mathcal{C}_0, \mathcal{J}_0)[m]$$

induced by (4.2). This means that liftings of torsors are unique up to p-torsion.

Proof. From the sequence (4.2) we know that the obstruction to lifting the cohomology class associated to a $\mathcal{X}_0$ lies inside $H^2(\mathcal{C}_0, \mathrm{Lie}(\mathcal{J}_0/\mathcal{C}_0)) \otimes I$. Note that since $\mathrm{Lie}(\mathcal{J}_0/\mathcal{C}_0)$ is a coherent $\mathscr{O}_{\mathcal{C}_0}$-modul, we can compute its cohomology with respect to the Zariski topology. Since $\mathcal{C}_0$ is one-dimensional, this group is zero.

Once we have lifted the cohomology class, we have to answer the question, whether it is associated to a representable $\mathcal{J}$-torsor. By [Mil80, Theorem 4.3 (e)] this will be the case if it is torsion. We claim that $H^1(\mathcal{C}, \mathcal{J})$ is torsion: Since $H^1(C, J)$ is torsion, it is enough to show that $H^1(\mathcal{C}_0, \mathrm{Lie}(\mathcal{J}_0/\mathcal{C}_0)) \otimes I$ is torsion, then the assertion will follow by induction. But the former group is a Λ-module, and Λ itself is annihilated by some power of p.

The second statement follows now directly by taking m-torsion in (4.2). □

4.2. *Remark.* In the case where the base is zero dimensional, one recovers the well known fact that the Tate-Šafarevič group of an elliptic curve over a complete local ring with algebraically closed residue field is zero, since the first cohomology of the Lie algebra vanishes.

We want to rephrase the above proposition in the language of deformation functors. For that purpose, we define two deformation functors associated to an elliptic bundle $X \to C$ over k.

4.1. **Definition.** By a deformation of X over some $\Lambda \in \mathcal{A}$, we mean a pair $(\mathcal{X}, \epsilon)$, where $\mathcal{X}$ is a flat scheme over $\mathrm{Spec}(\Lambda)$ and ϵ is an isomorphisms $\epsilon\colon \mathcal{X} \otimes_\Lambda k \simeq X$.

Let $\mathcal{D}ef_X \colon \mathcal{A} \to (Sets)$ denote the functor, which sends $\Lambda \in \mathcal{A}lg$ to the set of isomorphism classes of deformations of X/C.

By a *deformation of a fibration* X/C, we understand a deformation $(\mathcal{X}, \epsilon)$ of X together with a map $\mathcal{X} \to \mathcal{C}$, such that the isomorphism ϵ is in fact an isomorphism of C schemes.

The functor of deformation of X as fibration is denoted by $\mathcal{F}ib_{X/C} \colon \mathcal{A} \to (Sets)$.

Two deformations of $\mathcal{X}/\mathcal{C}$ and $\mathcal{X}'/\mathcal{C}$ are called *isomorphic* if there exists an isomorphisms of deformations, which is also an isomorphisms of $\mathcal{C}$-schemes.

Let J/C be the Jacobian fibration associated to X/C. Denote its zero section by $\epsilon_0 \colon C \to J$. We define the $\mathcal{J}ac_{J/C}$ the functor of liftings of J/C with a fixed lifting ϵ of ϵ_0. For an element $(\mathcal{J}, \epsilon)$ of $\mathcal{J}ac_{J/C}$ note that a different choice of ϵ leads to an isomorphic element of $\mathcal{J}ac_{J/C}$. Thus we view $\mathcal{J}ac_{J/C}$ as a subfunctor of $\mathcal{F}ib_{J/C}$; i.e. the subfunctor of those deformations admitting a lifting ϵ_0.

We get a natural map $\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}$ as follows: For a deformation $\mathcal{X}/\mathcal{C}$ (not necessarily having a section) we consider the zero component of its Picard scheme. Since k is of characterstic p, we can always lift an appropriate p^{th} power of a relative ample line bundle of $X \to C$. Therefore the representabilty of $\mathrm{Pic}_{X/C}$ follows from:

Theorem (Theorem 4.8 [Kle05])**.** *Let $f \colon Z \to S$ be projective and flat and have integral geometric fibres. Then $Pic_{Z/S}$ is representable by a separated S-scheme.*

Since $\mathcal{X}/\mathcal{C}$ is a relative curve, the Picard scheme will be smooth, and the zero component $\mathrm{Pic}^0_{\mathcal{X}/\mathcal{C}}$ is a smooth elliptic scheme over $\mathcal{C}$. Now there is a natural action of $\mathrm{Pic}^0_{\mathcal{X}/\mathcal{C}}$ on $\mathcal{X}/\mathcal{C}$ coming from the isomorphism

$$\mathrm{Pic}^1_{\mathcal{X}/\mathcal{C}} \simeq \mathcal{X}/\mathcal{C}.$$

We define the natural map $\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}$ by sending the fibration $\mathcal{X}/\mathcal{C}$ to $\mathrm{Pic}^0_{\mathcal{X}/\mathcal{C}}$. In this language, Proposition 4.1 now becomes the first part of our main theorem:

4.3. **Theorem.** *The map of functors $\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}$ is formally smooth. Furthermore we have*

$$\dim(\mathcal{F}ib_{X/C}(k[\epsilon])) = \dim(\mathcal{J}ac_{X/C}(k[\epsilon])) + h^1(C, \mathrm{Lie}(J/C)).$$

Proof. Recall that $\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}$ is formally smooth if for a surjection $\Lambda \twoheadrightarrow \Lambda_0$ in $\mathcal{A}lg$, which can be assumed to be small, the induced map

$$\mathcal{F}ib_{X/C}(\Lambda) \to \mathcal{F}ib_{J/C}(\Lambda_0) \times_{\mathcal{J}ac_{J/C}(\Lambda_0)} \mathcal{J}ac_{J/C}(\Lambda)$$

is surjective. However, this follows directly from Proposition 4.1 applied to every element $\mathcal{J}$ of $\mathcal{J}ac_{J/C}(\Lambda)$ with reduction $\mathcal{J}_0$ over Λ_0 and a $\mathcal{J}_0$ torsor $\mathcal{X}_0$.

To prove the statement about the tangent space dimensions, first note that $\mathcal{F}ib_{X/C}$ fullfills the Schlesinger criteria and carries therefore a vector space structure on its tangent space. We are going to determine the kernel of the following linear map

$$\mathcal{F}ib_{X/C}(k[\epsilon]) \to \mathcal{J}ac_{J/C}(k[\epsilon]),$$

which consists of torsors under $J \otimes k[\epsilon]$. To determine this group, we use again (4.2). The first term vanishes, since every section $C \to J$ lifts to the trivial deformation. Hence the kernel is given by $H^1(C, \mathrm{Lie}(J)) \otimes I$. □

5. Elliptic bundles of Kodaira dimension one

Let $f\colon X \to C$ be an elliptic bundle of Kodaira dimension one over k. We show that every deformation of X admits a fibration, which lifts the unique elliptic fibration on X. In other words:

5.1. **Theorem.** *For an elliptic bundle X of Kodaira dimension one, the inclusion of deformation functors $\mathcal{F}ib_{X/C} \to \mathcal{D}ef_X$ is an isomorphism.*

First, we need an estimate for $h^1(X, \Theta_X)$.

5.2. **Lemma.** *Denote by $g \geq 2$ the genus of C, and set $\mathscr{L} = R^1 f_* \mathscr{O}_X$. We get*

$$h^1(X, \Theta_X) \leq g - 1 + h^0(C, \mathscr{L}) + h^0(C, \mathscr{L}^2) + 3g - 3. \tag{5.1}$$

If X is Jacobian, we get equality in (5.1).

Proof. Since f is smooth, we have an exact sequence

$$0 \to \Theta_{X/C} \to \Theta_X \to f^*\Theta_C \to 0. \tag{5.2}$$

This gives rise to an exact sequence of cohomology groups

$$H^1(X, \Theta_{X/C}) \to H^1(X, \Theta_X) \to H^1(X, f^*\Theta_C).$$

Thus $h^1(X, \Theta_X) \leq h^1(X, f^*\Theta_C) + h^1(X, \Theta_{X/C})$. We claim that $\Theta_{X/C}$ is isomorphic to $f^*\mathscr{L}$: This follows from the canonical bundle formular

$$\omega_X \simeq f^*(\mathscr{L}^{-1} \otimes \omega_C)$$

(see [BM77, Theorem 2]), and from the expression

$$(\Theta_{X/C})^{-1} \simeq \omega_{X/C} \simeq \omega_X \otimes (f^*\omega_C)^{-1}.$$

To compute $h^1(X, \Theta_{X/C})$ we use the Leray spectral sequence and the projection formula, which yields

$$0 \to H^1(C, \mathscr{L}) \to H^1(X, \Theta_{X/C}) \to H^0(C, \underbrace{R^1 f_* f^* \mathscr{L}}_{\simeq \mathscr{L}^{\otimes 2}}) \to 0.$$

By Riemann-Roch we get $h^1(C, \mathscr{L}) = g - 1 + h^0(C, \mathscr{L})$. Thus

$$h^1(X, \Theta_{X/C}) = g - 1 + h^0(C, \mathscr{L}) + h^0(C, \mathscr{L}^{\otimes 2}).$$

For $h^1(X, f^*\Theta_C)$ we obtain with the same approach

$$0 \to H^1(C, \Theta_C) \to H^1(X, f^*\Theta_C) \to H^0(C, \mathscr{L} \otimes \Theta_C) \to 0.$$

Since $g > 1$, the last term vanishes and we get $h^1(X, f^*\Theta_C) = 3g - 3$.

In the Jacobian case let $s\colon C \to X$ denote the section. The natural map $f^*\Omega^1_C \to \Omega^1_X$ has a global splitting given locally by $d(g) \mapsto d(s^*g) \otimes 1$. Now, dualizing yields a splitting of (5.2). □

Next, we show the surjectivity of the inclusion $\mathcal{F}ib_{X/C} \to \mathcal{D}ef_X$.

5.3. **Proposition.** *Let Λ be an object of $\mathcal{A}lg$. Every deformation $\mathcal{X} \in \mathcal{D}ef_X(\Lambda)$ of the total space of X admits a lifting of the fibration on X; in other words $\mathcal{X} \in \mathcal{F}ib_{X/C}(\Lambda)$.*

Proof. Denote by J the Jacobian of X. By Proposition 3.2, we know that there is an étale Galois covering $C' \to C$ with group Γ, such that $J' = J \times_C C' = E \times_k C'$, for some elliptic curve over k. Since forming Pic^0 commutes with base change, the Jacobian associated to the fibration $X' = X \times_C C'$ will be J'. We denote by $\mathcal{X}' \to \mathcal{X}$ the unique lifting of $X' \to X$.

We claim that $\mathcal{X}'$ admits an elliptic fibration. To see this, we show that the deformation functors $\mathcal{F}ib_{X'/C'}$ and $\mathcal{D}ef_{J'/C}$ associated with X' are isomorphic. Since

J' has unobstructed deformations by Corollary 3.3, we conclude by Proposition 4.1, that $\mathcal{F}ib_{X'/C}$ is unobstructed as well. It remains to show that

$$h^1(X', \Theta_{X'}) = \dim(\mathcal{F}ib_{X'/C}(k[\epsilon])).$$

Let g denote the genus of C. We have $h^1(X', \Theta_{X'}) \leq 4g - 2$ by Lemma 5.2. As for $\mathcal{F}ib_{X'/C}(k[\epsilon])$, we have $(3g-3)+1$ dimensions coming from the functor of Jacobian deformations of J': Namely $3g-3$ from the deformations of the C', and one dimension coming from E. The first cohomology of the Lie algebra $\mathscr{O}_{C'}$ of J' gives g additional dimensions.

Now, we come back to our deformation $\mathcal{X}$ of X. We claim that the fibration $g\colon \mathcal{X}' \to C'$ descends to $\mathcal{X}$. For this we have to show that for each $\sigma \in \Gamma$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\sigma} & \mathcal{X}' \\ \downarrow & & \downarrow \\ C & \xrightarrow{\sigma} & C \end{array}$$

However, this follows because the elliptic fibration on $\mathcal{X}'_1$ is unique: First note, that this is the case on the reduction X' because of Kodaira dimension one. For a given first order deformation, the tangent space of the functor of deformations of $g\colon X' \to C'$ is $H^0(X', g^*\Theta_{C'})$. However, this vector space is trivial since the dual $g^*\Omega^1_{C'}$ of $g^*\Theta_{C'}$ has a non zero global section, because $g(C') \geq 2$. Now our claim follows by induction on the length of Λ. □

6. Deformations of bielliptic surfaces

As mentioned in the introduction, a surface X over k is called bielliptic, if it is of Kodaira dimension zero with invariants $b_1 = b_2 = 2$. Directly from the invariants, we get that the Albanese of X is an elliptic curve. The associated map $f\colon X \to \mathrm{Alb}(X)$ is a smooth elliptic fibration; see [BM77, Proposition 5].

To keep the presentation streamlined, we first consider the cases where f has a section. From Proposition 3.2 we know that X is given by a quotient

$$(E \times F)/\Gamma$$

where $F \to \mathrm{Alb}(X)$ is an étale Galois covering of group Γ (i.e. an étale isogeny) and Γ acts on E fixing the zero section. Note that this action cannot be trivial, for otherwise X would be an abelian surface. Without loss of generality, we assume the action of Γ on E faithful. Since the fundamental group of an elliptic curve is abelian, Γ has to be abelian too. It follows that Γ equals $\mathbb{Z}/d\mathbb{Z}$, where $d \in \{2, 3, 4, 6\}$. We fix the image of a generator of Γ in $\mathrm{Aut}_0(E)$, and denote it by ω. As a first step, we calculate some invariants of X depending on p and d which are important for the deformation behaviour of X.

6.1. **Lemma.** *If d is not a power of p and $d \neq 2$ we have*

$$h^0(X, \Theta_X) = 1, \quad h^1(X, \Theta_X) = 1, \quad h^2(X, \Theta_X) = 0, \quad h^1(\mathrm{Alb}(X), \mathrm{Lie}(X)) = 0.$$

If $d = 2$ and $p \neq 2$ we get

$$h^0(X, \Theta_X) = 1, \quad h^1(X, \Theta_X) = 2, \quad h^2(X, \Theta_X) = 1, \quad h^1(\mathrm{Alb}(X), \mathrm{Lie}(X)) = 0.$$

Whereas if d is a power of p it holds

$$h^0(X, \Theta_X) = 2, \quad h^1(X, \Theta_X) = 4, \quad h^2(X, \Theta_X) = 2, \quad h^1(\mathrm{Alb}(X), \mathrm{Lie}(X)) = 1.$$

Proof. Set $C = \mathrm{Alb}(X)$. Since f is smooth, we have an exact sequence

$$0 \to \Theta_{X/C} \to \Theta_X \to f^*\Theta_C \to 0.$$

On the covering $E \times F$ of X the corresponding sequence is split, and since the action of Γ is diagonal, the splitting descends to X. Therefore Θ_X decomposes as

$$\Theta_X \simeq \Theta_{X/C} \oplus f^*\Theta_C.$$

Since C is an elliptic curve, we find $f^*\Theta_C \simeq \mathscr{O}_X$. As for $\Theta_{X/C}$, it will be a torsion line bundle of order l equal to the order of the induced action of Γ on Θ_E. To see this, note that Γ acts trivially on $\Theta_E^{\otimes l}$ because the induced action is by roots of unity. Thus a section of $\Theta_E^{\otimes l}$ will descend to a section of $\Theta_{X/C}$.

If d is a power of p, this action has to be trivial, since $\mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m) = 0$.

To determine the action in general, note that we have $\mathrm{Lie}(E) = \mathrm{Lie}(E[p]) = \mathrm{Lie}(E[p]^0)$. We set $H = E[p]^0$. It will be either α_p or μ_p. These are group schemes of height-one, so the map given by the Lie functor $\mathrm{Aut}(H) \to \mathrm{Aut}(\mathrm{Lie}(H))$ is injective. In fact, it will be an isomorphism if we restrict to maps of p-Lie algebras (see [Mum70, Section 14]).

The group scheme H is of rank p, so if $p > 4$ we get $\mathrm{ord}(\omega) = \mathrm{ord}(\omega|_H)$ by rigidity ([KM85, Corollary 2.7.3]). If $p = 3$ and $k = 2$, we know that ω will act on $\mathrm{Lie}(E)$ as involution. If $d = 4$ the same argument applies to ω^2.

If $p = 2$ and $k = 3$ we have $H = \alpha_2$ since $j(E) = 0$. If ω induces the identity on H, the associated trace map $\mathrm{tr}_\omega = \mathrm{Id} + \omega + \omega^2$ would give multiplication by 3, which is an isomorphisms on H. But we know that tr_ω is the zero map on E (see Lemma 6.5 below).

To sum up this discussion, we get that $\mathrm{ord}(\Theta_{X/C}) = l$ where $d = lp^n$ with l prime to p. Now it easy to calculate the invariants. Denote by $\epsilon\colon C \to X$ the zero section of X. We have $\mathscr{L} = \mathrm{Lie}(X) = \epsilon^*\Theta_{X/C}$, and since $f^*\mathrm{Lie}(X) \simeq \Theta_{X/C}$ it follows that

$$\mathrm{ord}(\mathrm{Lie}(X)) = \mathrm{ord}(\Theta_{X/C}).$$

The statement about the cohomology of $\mathrm{Lie}(X)$ follows, since it is a line bundle of degree zero and therefore

$$h^1(X, \mathrm{Lie}(X)) = h^0(X, \mathrm{Lie}(X)).$$

However, the last term is not zero if and only if $\mathrm{Lie}(X)$ is trivial.

To compute $h^1(X, \Theta_X)$ we treat both sumands separately. We have $\Theta_{X/C} \simeq f^*R^1 f_*\mathscr{O}_X \simeq \mathrm{Lie}(X)$. By the projection formula and the Leray spectral sequence we get

$$0 \to H^1(C, \mathscr{L}) \to H^1(X, \Theta_{X/C}) \to H^0(C, \underbrace{R^1 f_* f^* \mathscr{L}}_{\simeq \mathscr{L}^{\otimes 2}}) \to 0.$$

Thus we get:

$$h^1(X, \Theta_{X/C}) = \begin{cases} 0 & \text{if } l > 2 \\ 1 & \text{if } l = 2 \\ 0 & \text{if } l = 1 \end{cases}$$

For $h^1(X, f^*\Theta_C)$ we obtain similarly:

$$0 \to H^1(C, \Theta_C) \to H^1(X, f^*\Theta_C) \to H^0(C, \mathscr{L} \otimes \Theta_C) \to 0$$

Because $\Theta_C \simeq \mathscr{O}_C$ we find $h^1(X, f^*\Theta_C) = 2$ if $\mathscr{L}$ is trivial and $h^1(X, f^*\Theta_C) = 1$ otherwise.

This proves the statement about $h^1(X, \Theta_X)$. The remaining cohomology groups are calculated easily using duality. □

6.1. **The versal families.** Let $X = (E \times F)/G$ over k be a Jacobian bielliptic surface. First, we to study the deformation functor $\mathcal{J}ac_{X/C}$.

By Proposition 3.2 we know the structure of Jacobian deformations of X. They will be of the form $(\mathcal{E} \times \mathcal{F})/\Gamma$. Here, $\mathcal{E}$ is a deformation of E extending the automorphism ω, and we are going to denote the deformation functor of such pairs by (E, ω). Likewise, $\mathcal{F}$ is a deformation of F with a torsion point lifting the point of F which appears in the definition of the action of Γ, and we denote the deformation functor of such pairs by (F, c).

The functor $\mathcal{J}ac_{X/C}$ is isomorphic to the product of the deformation functors

$$(E, \omega) \times (F, c).$$

To write down a versal family for $\mathcal{J}ac_{X/C}$, we treat the problem separately for both factors.

6.1.1. *Deforming elliptic curves with automorphisms.* Let $(\mathcal{E}^{univ}, \omega) \to \operatorname{Spec}(R)$ be the universal deformation of E along with its automorphism ω. This functor is indeed pro-representable since if a lifting of ω exists for a given deformation of E, then it is unique.

If $\operatorname{ord}(\omega) = 2$, then ω is the involution, which obviously extends to any deformation of E. Hence in that case $R = W[[j]]$.

If $\operatorname{ord}(\omega) > 2$ then the j-invariant of $\mathcal{E}$ is either 0 or 1728. If the order is prime to p, there are no obstruction against lifting (E, ω), thus $R = W$.

We treat the remaining cases. First, assume $p = 2$ and $d = \operatorname{ord}(\omega) = 4$. We know from [JLR09, Lemma 1.1] that there is no elliptic curve over W, with j-invariant 1728 and good reduction. This means we have to pass to a ramified extension of W. We will work over $R = W[i]$ where i is a primitive forth root of unity. The following curve $\mathcal{E}_2$ is taken from [JLR09, §2.A]

$$y^2 + (-i + 1)xy - iy = x^3 - ix^2.$$

It has $j = 1728$ and Discriminant $\Delta = 11 - 2i$, and is therefore of good reduction.

For $p = 3$ and $d = 3$, again by [JLR09], there will be no elliptic curve over W with j-invariant 0 and good reduction. So let $R = W[\pi]$, where $\pi^2 = 3$. Consider the elliptic curve $\mathcal{E}_3$ given by the Weierstraß equation

$$y^2 = x^3 + \pi x^2 + x,$$

whose j-invariant is 0 and whose discriminant is $\Delta = -16$. In particular, it has good reduction.

In both cases ($p = 2$ or 3), the curve $\mathcal{E}_p$ has an automorphism of order four or three respectively, since on the generic fibre, automorphisms are given by the action of certain roots of unity, and we have chosen the baserings in such a way, that they contain the necessary roots. An automorphism of the generic fibre extends to the entire family, and its order will not change after passing to the reduction, as can be seen by considering an étale torsion subscheme of sufficiently high order.

We claim that the elliptic curves over the rings constructed above are the universal families for the deformation problem (E, ω). This follows from the fact that the respective base rings are the smallest possible extensions of W over which the deformation problem can be solved, and from the fact that an elliptic curve over a strictly henselian ring is determind by its j-invariant.

6.1.2. *Deforming elliptic curves with torsion points.* Now we treat the second factor. If p does not divide d, then a d-torsion point lifts uniquely to any deformation of F. Therefore (F, c) is pro-represented by $W[[j]]$.

Assume now, that p does divide d. By the above, we can assume $d = p^n$. Since F is ordinary, we can use the theory of *Serre-Tate local moduli.* For the special case of ordinary elliptic curves, see [KM85, 8.9]. By this theory, we can represent the deformation functor of F by a pair

$$\mathcal{F}^{univ} \to \operatorname{Spec}(W[[q-1]])$$

satisfying the following property: For a complete local $W[[q-1]]$-algebra Λ, consider the pullback

$$\mathcal{F} = \mathcal{F}^{univ} \otimes_{W[[q-1]]} \Lambda.$$

By $\mathbb{Z}[q, q^{-1}] \to W[[q-1] \to \Lambda$ we make Λ into a $\mathbb{Z}[q, q^{-1}]$-algebra. There exists a universal group scheme T over $\mathbb{Z}[q, q^{-1}]$ defined in [KM85, 8.7], such that

$$\mathcal{F}[p^\infty] \simeq T[p^\infty] \otimes_{\mathbb{Z}[q,q^{-1}]} \Lambda.$$

The explicit description of T in [KM85, 8.7] implies that the sequence

$$0 \to \mu_{p^n} \to T[p^n] \otimes \Lambda \to \mathbb{Z}/p^n\mathbb{Z} \to 0 \tag{6.1}$$

is split if and only if the image of q in Λ has a p^n-th root. However, (6.1) is split if and only if c lifts to $\mathcal{F}$.

We conclude that $W[[q-1]][\sqrt[p^n]{q}]$ is a versal hull of the functor (F, c).

6.2. **Proposition.** *The functor $\mathcal{J}ac_{X/C}$ has R for a versal hull, where R is given in the table below:*

	$p=2$	$p=3$	$p>3$
$k=2$	$W[[j_E]] \otimes W[[q-1]][\sqrt[2]{q}]$	$W[[j_E]] \otimes W[[j_F]]$	$W[[j_E]] \otimes W[[j_F]]$
$k=3$	$W[[j_E]] \otimes W$	$W[\pi] \otimes W[[q-1]][\sqrt[3]{q}]$	$W \otimes W[[j_F]]$
$k=4$	$W[i] \otimes W[[q-1]][\sqrt[4]{q}]$	$W \otimes W[[j_F]]$	$W \otimes W[[j_F]]$
$k=6$	$W \otimes W[[q-1]][\sqrt[2]{q}]$	$W[\pi] \otimes W[[q-1]][\sqrt[3]{q}]$	$W \otimes W[[j_F]]$

It is easy to read off and interpret the dimension of the tangent space of the deformation functor. For example in the case where $p=3$ and $d=3$ we have $\dim(\operatorname{Hom}(W[\pi] \otimes W[\sqrt[3]{j_E}], k[\epsilon])) = 3$. The parameter j_F gives one dimension, and the rest is due to relations, coming from obstructions. As explained before, we have $h^1(X, \Theta_X) = 4$ in this case, so there still is one dimension missing.

To account for this missing dimension, we have to study all deformations of X, not just the Jacobian ones. This is settled by Theorem 4.3. Observe that in all the cases, we have

$$h^1(X, \Theta_X) - \dim(\mathcal{J}ac_{X/C}(k[\epsilon])) = h^1(C, \operatorname{Lie}(X)).$$

Therefore $\dim(\mathcal{F}ib(k[\epsilon])) = h^1(X, \Theta_X)$, and its makes sense to ask if the absolute deformation functor of X is isomorphic to $\mathcal{F}ib_{X/C}$. In the next section, we will see that this is indeed the case.

6.2. **Classification of deformations.** The most important step to classify deformations of bielliptic surfaces is to show that for a bielliptic surface X/C over k the functors $\mathcal{F}ib_{X/C}$ and $\mathcal{D}ef_X$ are isomorphic.

Denote by $J \to C$ the Jacobian of $X \to C$. If d is not a power of p, the claim follows already, since in that case $\mathcal{J}ac_{J/C}$ is unobstructed, and has the right tangent dimension. Hence we get a chain of isomorphisms

$$\mathcal{J}ac_{X/C} \simeq \mathcal{F}ib_{X/C} \simeq \mathcal{D}ef_X .$$

In the case where d is a power of p, we have to work with the étale covering of X. This is more difficult than in the Kodaira dimension one case, because the étale cover of the reduction is an elliptic abelian surface and not every deformation of the covering admits a fibration.

To understand the deformation theory of abelian surfaces, we use p-divisible groups. For the readers convenience, we repeat some basic definitions and facts:

Let p be a prime number, and let S be a scheme. A sheaf of groups for the fppf-topology is called a *p-divisible group*, if G is p-divisible and p-primary, i.e.

$$G = \varinjlim G[p^n]$$

and the groups $G[p^n]$ are finite flat group scheme over S (see [Gro74] where p-divisible groups go by the name "Barsotti-Tate groups"). The main examples, and those which we are in fact interested in, are p-divisible groups associated with abelian schemes. For an abelian S-scheme A, we set

$$A[p^\infty] = \varinjlim A[p^n].$$

The deformation theory of abelian schemes is controlled by p-divisible groups. To be precise, let R be a ring in which $p^N = 0$. For a nilpotent ideal $I \subset R$ we define the category $\mathcal{T}$ of triples:

$$(A, G, \epsilon)$$

where A is an abelian scheme over R/I, $\mathcal{G}$ is a p-divisible group over R and ϵ an isomorphisms $\mathcal{G} \otimes_R R/I \simeq A[p^\infty]$. Now we have the theorem of Serre and Tate:

6.3. **Theorem** (Theorem 1.2.1 [Kat81])**.** *There is an equivalence between $\mathcal{T}$ and the category of abelian schemes over R given by*

$$\mathcal{A} \mapsto (\mathcal{A} \otimes_R R/I, \mathcal{A}[p^\infty],\ \textit{natural}\ \epsilon).$$

We will use the following statement, to understand the lifting behavior of morphisms of the latter:

6.4. **Lemma** (Lemma 1.1.3 [Kat81])**.** *Let $\mathcal{G}$ and $\mathcal{H}$ be p-divisible groups over R. Assume $I^{\nu+1} = 0$. Let G and H denote their restrictions to* $\operatorname{Spec}(R/I)$*. Then the following holds:*

(i) *The groups $\operatorname{Hom}_R(\mathcal{G}, \mathcal{H})$ and $\operatorname{Hom}_{R/I}(G, H)$ have no p-torsion.*
(ii) *The reduction map $\operatorname{Hom}_R(\mathcal{G}, \mathcal{H}) \to \operatorname{Hom}_{R/I}(G, H)$ is injective.*
(iii) *For any homomorphism $f\colon G \to H$ there exists a unique homomorphism ϕ_ν lifting $[p^\nu] \circ f$.*
(iv) *In order for f to lift to an homomorphism $f\colon \mathcal{G} \to \mathcal{H}$, it is necessary and sufficient for the homomorphism ϕ_ν to annihilate $\mathcal{G}[p^\nu]$.*

In the course of the proof, we will use the following lemma:

6.5. **Lemma.** *Let $\mathcal{E}$ be an elliptic scheme over a base scheme $\mathcal{S}$. Let Ω be an automorphism of $\mathcal{E}$ of order d. We consider the trace map:*

$$\operatorname{tr}_\Omega = \operatorname{Id} + \Omega + \cdots + \Omega^{d-1}.$$

Then tr_Ω gives the zero map on $\mathcal{E}$.

Proof. We can prove the statement fibrewise. So assume that S it the spectrum of a field. Now for any $\mathcal{S}$-scheme T, and a T-valued point $x \in \mathcal{E}_\mathcal{T}(\mathcal{T})$ we find that

$$\operatorname{tr}_\Omega(x) = \operatorname{tr}_\Omega(\Omega(x)).$$

In other words: The orbits of Ω are contained in the fibres of tr_Ω. This means in particular, that tr_Ω factors over the quotient scheme $\mathcal{E}/(\mathbb{Z}/d\mathbb{Z})$, where the action is given by Ω. However, since Ω fixes the zero-section, this quotient is isomorphic to $\mathbb{P}^1_\mathcal{S}$.

Now the conclusion follows, because there is no non-constant map $\mathbb{P}^1_\mathcal{S} \to \mathcal{E}$. □

6.6. **Proposition.** *Let X be a Jacobian bielliptic surface over k with $d = p^n$. Denote by $f\colon X \to C$ the smooth elliptic fibration. Then f extends to any deformation $\mathcal{X}$ of X over $\Lambda \in \mathcal{A}lg$.*

Proof. As in the case of Kodaira dimension one, we start with the étale cover $A = E \times F$ of X. Again, we get a diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \end{array}$$

where the right hand column is the unique lifting of the left. By [MFK94, Theorem 6.14] we can give $\mathcal{A}$ a structure of an abelian scheme, extending the group structure on A. Now the strategy is as follows: First, we show that $\mathcal{A}$ has an automorphism Ω comming from ω. Then we study the action of Ω on the p-divisible group $\mathcal{A}[p^\infty]$, and use the trace map defined by Ω to lift the projection $E[p^\infty] \times F[p^\infty] \to F[p^\infty]$. This lifting will descends to the desired lifting of the fibration f on X.

The covering $\mathcal{A} \to \mathcal{X}$ is Galois with a group Γ isomorphic to $\mathrm{Gal}(A/X)$. Denote by σ a generator of Γ. We study its action on $\mathcal{A}$: Set $c = \sigma(0) \in \mathcal{A}(\mathrm{Spec}(\Lambda))$ and denote by t_{-c} the morphisms given by translation with $-c$. We set

$$\Omega = \sigma \circ t_{-c} \quad \text{and} \quad \Omega' = t_{-c} \circ \sigma.$$

Both maps fix the zero section of $\mathcal{A}$, and are therefore group automorphisms of $\mathcal{A}$. Furthermore, they lift the automorphism $\mathrm{Id} \times \omega$ of $E \times F$, which implies $\Omega = \Omega'$ since liftings of automorphisms are unique.

It follows now, that Ω and t_c commute, and since σ and Ω are of order d, we get that c is torsion point of order d, lifting the action of Γ by translation on F.

To proceed with the proof, we pass to the category of p-divisible groups, as explained in Theorem 6.3.

Our aim is to lift the second projection $\mathrm{pr}_2 \colon E[p^\infty] \times F[p^\infty] \to F[p^\infty]$. We know there exists some integer N such that there exists a unique lift ϕ_N of $[N] \circ \underline{\mathrm{pr}_2}$ (Lemma 6.4). We compare ϕ_N with the trace tr_Ω defined by Ω. The restriction $\overline{\mathrm{tr}_\Omega}$ of tr_Ω to $A[p^\infty]$ gives the map

$$[d] \circ \mathrm{pr}_2 \colon A[p^\infty] \to F[p^\infty] = \mathrm{Im}(\overline{\mathrm{tr}_\Omega}),$$

because tr_Ω gives multiplication with $[d]$ on the factor $F[p^\infty]$ and the zero map on the factor $E[p^\infty]$ (see Lemma 6.5). Now, we get that $[d] \circ \phi_N$ is a lift of $[N] \circ \overline{\mathrm{tr}_\Omega}$. Since lifts of endomorphisms are unique, it follows

$$[d] \circ \phi_N = [N] \circ \mathrm{tr}_\Omega\,.$$

Factoring out by $\mathcal{A}[N]$, we see that tr_Ω is a lift of $[d] \circ [\mathrm{pr}_2]$. It remains to show that $\mathcal{A}[d]$ lies in the kernel of tr_Ω.

To see this, we consider the exact sequence of finite flat group schemes

$$0 \to \mathcal{A}[d]^0 \to \mathcal{A}[d] \to \mathcal{A}[d]^{et} \to 0.$$

We first show $\mathrm{tr}_\Omega(\mathcal{A}[d]^0) = 0$. Again we have an exact sequence

$$0 \to \mathcal{A}[d]^{mult} \to \mathcal{A}[d]^0 \to \mathcal{A}[d]^{bi} \to 0.$$

The outer groups denote the multiplicative part and the biinfinitesimal part respectively. The category of multiplicative groups schemes is dual to the category of étale group schemes via Cartier duality - thus endomorphisms lift uniquely, and we get $\mathrm{tr}_\Omega(\mathcal{A}[d]^{mult}) = 0$. Now, we consider the sequence of p-divisible groups

$$0 \to \mathcal{A}[p^\infty]^{mult} \to \mathcal{A}[p^\infty]^0 \to \mathcal{A}[p^\infty]^{bi} \to 0.$$

Since Ω maps $\mathcal{A}[p^\infty]^{mult}$ into itself, we get an induced action of Ω on $\mathcal{A}[p^\infty]^{bi}$, and in particular, tr_Ω descends to $\mathcal{A}[p^\infty]^{bi}$. If this group is not trivial, it is a lift of $E[p^\infty]$ on which tr_Ω is zero. Again by uniqueness of lifts, we get $\mathrm{tr}_\Omega(\mathcal{A}[d]^{bi}) = 0$.

We have seen that $\mathrm{tr}_\Omega(\mathcal{A}[d]^0) = 0$, and it remains to show $\mathrm{tr}_\Omega(\mathcal{A}[k]^{et}) = 0$. However, this is clear, since we deal with étale group schemes. We conclude that pr_2 extends to $\mathcal{X}$. □

So far, we have treated only Jacobian bielliptic surfaces. But the non-Jacobian cases are mostly trivial. Consulting the table of bielliptic surfaces in [BM77], we see that the Tate-Šafarevič group is trivial if the associated Jacobian has obstructed deformations, except in one case in charactertic two.

To construct this surface, let E and F be ordinary elliptic curves over k with $p = 2$. We set $A = (E \times F)/\mu_2$, where μ_2 is the subgroup scheme embedded diagonally into

$$(E \times F)[2]^0 \simeq \mu_2 \times \mu_2.$$

The quotient A is an abelian surface which does not split into a product.

Let c be a non trivial 2-torsion point of F. The action on $E \times F$, given by $(x, y) \mapsto (x+c, -y)$, commutes with the diagonal action of μ_2 and thereby descends to a $\mathbb{Z}/2\mathbb{Z}$ action on A. The bielliptic surface X we are interested in is now given by $A/(\mathbb{Z}/2\mathbb{Z})$. The Jacobian of X is clearly $J = (E \times F)/(\mathbb{Z}/2\mathbb{Z})$.

Now let $\mathcal{X}$ be a deformation of X. Once more we have a diagram:

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \end{array}$$

We claim that $\mathcal{A}$ admits a lifting of the elliptic fibration $f\colon A \to F/\mu_2$: We have an exact sequence

$$0 \to \mathcal{A}[p^\infty]^{tor} \to \mathcal{A}[p^\infty] \to \mathcal{A}[p^\infty]^{et} \to 0. \tag{6.2}$$

The morphism $A[p^\infty]^{et} \to F[p^\infty]^{et}$ induced by f lifts uniquely to

$$\varphi\colon \mathcal{A}[p^\infty]^{et} \to \mathcal{F}[p^\infty]^{et},$$

since we are dealing with étale group schemes. Denote by $\mathcal{B}$ the p-divisible group obtained by pushout of (6.2) along φ. We still have $\mathcal{A}[p^\infty]^{tor} \subset \mathcal{B}$ and inside $\mathcal{A}[p^\infty]^{tor}$ we have the kernel of the unique lift of $A^{tor} \to F[p^\infty]^{tor}$. Dividing out $\mathcal{B}$ by that kernel we obtain a lifting of f.

As in the proof of Theorem 6.6, we see that f descends to $\mathcal{X}$. Therefore $\mathcal{X}$ is elliptic. To sum up, we have the following theorem:

6.7. **Theorem.** *Every deformation X of a bielliptic surface X induces a lifting of the elliptic fibration $X \to C = \mathrm{Alb}(X)$.*

Next, we show that a versal deformation of a bielliptic surface is algebraizable.

6.8. **Proposition.** *Let X be a bielliptic surface over k. Denote by $\mathcal{X}^{vers} \to \mathrm{Spf}(R)$ a formal versal family of $\mathcal{D}ef_X$. Then there exists a projective scheme $\overline{\mathcal{X}}$ over R, such that $\mathcal{X}^{vers}$ is the completion of $\overline{\mathcal{X}}$ at the special fibre.*

Proof. For an arbitrary deformation $\mathcal{X}$ of X, denote by $\mathcal{A} \to \mathcal{X}$ the unique lifting of the abelian covering $A \to X$. In the proof of Proposition 6.6 we saw that the abelian scheme $\mathcal{A}$ has an automorphism Ω lifting the automorphism $\omega \times \mathrm{Id}$ of $E \times F$.

The automorphism of $\mathcal{X}$, obtained from Ω by descent, will again be denoted by Ω. Now Ω is a $\mathcal{C}$-automorphism of $\mathcal{X}$; i.e. its action is confined to the fibres of the fibration.

We claim that the fixed locus of Ω is flat over $\mathcal{C}$: Every closed point $x \in \mathcal{C}$ has an étale neighborhood $\mathcal{U} \to \mathcal{C}$, such that the the pullback

$$\mathcal{X}_\mathcal{U} = \mathcal{X} \times_\mathcal{C} \mathcal{U}$$

can be given the structure of an abelian scheme, in such a way that the base change of Ω to $\mathcal{X}_{\mathcal{U}}$ becomes a group automorphism. We consider the endomorphism $\Omega - \mathrm{Id}$ of $\mathcal{X}_{\mathcal{U}}$. It is a surjective map of abelian schemes, and therefore flat by ([MFK94, Lemma 6.12]). In particular its kernel, i.e the fix locus of Ω, is flat over $\mathcal{U}$.

Thus we have found a relative Cartier divisor of $\mathcal{X} \to \mathcal{C}$. Its degree can be computed on the reduction. It equals the order of the subgroup scheme of E fixed by ω. In particular it is positive, which means that $\mathcal{Z}$ is a relatively ample divisor for $\mathcal{X} \to \mathcal{C}$.

Now denote by $\mathcal{X}^{vers} \to \mathrm{Spf}(R)$ a versal family of $\mathcal{F}ib_{X/C}$. It is a formal scheme over the hull of the deformation functor $\mathcal{F}ib_{X/C}$, therefore admitting an elliptic fibration $F\colon \mathcal{X}^{vers} \to \mathcal{C}$ lifting $X \to C$. Denote by $\mathfrak{m}$ the maximal ideal of R , and set $X_n = \mathcal{X}^{vers} \otimes_R R/\mathfrak{m}^{n+1}$.

The construction of $\mathcal{Z}$ gives rise to a compatible system of relatively ample line bundles $\mathscr{O}_{X_n}(Z_n)$. Tensoring with the line bundle coming from the divisor of a fibre of $\mathcal{X}^{vers} \to \mathcal{C}$, we obtain a system of ample line bundles $\mathscr{H}_n$. Thus by Grothendieck's algebraization theorem [Ill05, Theorem 4.10], we conclude that $\mathcal{X}^{vers}$ is the completion of some projective scheme $\overline{\mathcal{X}^{vers}}$ over $\mathrm{Spec}(R)$. □

The above proposition helps us to answer another natural question: X is called bielliptic because it has two transversal elliptic fibrations: The smooth one, denoted by f, coming from the projection $E \times F \to F$ and a second one, denoted by g, with base curve $\mathbb{P}^1_k$ coming from $E \times F \to E$. We saw that the first fibration is preserved under deformation, but what about the second one?

6.9. **Proposition.** *Let X be a bielliptic fibration, then every deformation $\mathcal{X}$ of X extends both elliptic fibration.*

Proof. We are going to show that the versal deformation $\mathcal{X}^{vers} \to \mathrm{Spf}(R)$ admits an extension of g, then the claim follows by versality.

Denote by K the fraction field of R. We can use surface theory to analyze the generic fibre $\overline{\mathcal{X}}_K$ of the algebraization $\overline{\mathcal{X}}$ of $\mathcal{X}^{vers}$. Denote by $\mathscr{L} = \mathscr{O}_{\overline{\mathcal{X}}}(\mathcal{Z})$ the line bundle associated to the divisor $\mathcal{Z}$, constructed in the proof of Proposition 6.8. The canonical bundle of $\overline{\mathcal{X}}_K$ has self-intersection number 0. It follows that the line bundle $\mathscr{L}_K^{\otimes m}$, gives rise to an elliptic fibration $g'\colon \overline{\mathcal{X}}_K \to \mathbb{P}^1_K$, if we choose m sufficiently big [Băd01, Theorem 7.11].

Since $\overline{\mathcal{X}}$ is proper and normal, we can extend g' to a rational map $g'\colon \overline{\mathcal{X}} \to \mathbb{P}^1_R$ which is defined on a non-empty open subset intersecting the special fibre. Now, there are sections $s_1, s_2\colon \mathrm{Spec}(R) \to \mathbb{P}^1_R$ whose associated closed subschemes are disjoint and who do lie inside the image of g'. Taking the closures of the inverse images of those sections under g', we get two divisors $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\overline{\mathcal{X}}$ who have disjoint specializations on a non empty open subset of the special fibre (namely the locus where g' is defined).

We claim that their reductions G_1 and G_2 are irreducible (and hence disjoint): The group of divisors of X modulo numerical equivalence is generated by two classes F and G, where F is a fibre class of f and G is one of g. The intersection numbers are

$$F \cdot F = 0, \qquad F \cdot G > 0, \qquad G \cdot G = 0.$$

In particular, there are no effective divisors on X with negative self-intersection. It follows that the specialization of a curve of canonical type is again of canonical type. However, every curve of canonical type on X is irreducible, hence the claim follows.

Considering the global sections of $\mathscr{L}^{\otimes m}$ associated to the effective divisors $\mathcal{G}_1$ and $\mathcal{G}_2$, we find that $\mathscr{L}^{\otimes m}$ is globally generated. It follows that the map given by $\mathscr{L}^{\otimes m}$ is in fact a morphism, lifting g. □

We illustrate the theorem by looking at a special case: Let k be of characteristic three, and let X denote the Jacobian bielliptic surface of index $d = 3$ over k. What does the fibre $\mathcal{X}_\eta$ of the versal family of X over the generic point of the base look like? The smooth fibration with elliptic base curve does not have a section. The three sections which appear when we basechange with the algebraic closure of η do not descend to $\mathcal{X}_\eta$. Instead, we have a multi-section of degree three.

There is an explicit construction of a bielliptic surface with $d = 3$ over $\mathbb{Q}$, which shows the same behaviour. It was given in [BS03] as a counterexamples to the Hasse principle which cannot be explained by the Manin obstruction.

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