

DEFORMATIONS OF EQUIVELAR STANLEY-REISNER ABELIAN SURFACES

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ABSTRACT. The versal deformation of Stanley-Reisner schemes associated to equivelar triangulations of the torus is studied. The deformation space is defined by binomials and there is a toric smoothing component which I describe in terms of cones and lattices. Connections to moduli of abelian surfaces are considered. The case of the Möbius torus is especially nice and leads to a projective Calabi-Yau 3-fold with Euler number 6.

INTRODUCTION

Interesting versal deformation spaces in algebraic geometry are few and far between. They tend to be either smooth, i.e. the object to deform is unobstructed, or much too complicated to compute. In general the equations defining formal versal base spaces are not polynomials.

The purpose of this paper is to present an exception to the rule. In [AC09] we showed that triangulated surface manifolds with regular edge graph of degree 6, give Stanley-Reisner schemes with nicely presented formal versal deformations spaces defined by polynomials, in fact binomials. Such a surface is either a torus or a Klein bottle.

In this paper, the torus case is studied. Triangulations of (or more generally maps on) surfaces with regular edge graph are called equivelar. In the torus case they were only recently classified and counted by Brehm and Kühnel in [BK08]. The Stanley-Reisner schemes of such tori are all smoothable and they smooth to abelian surfaces.

To avoid non-algebraic abelian surfaces I will work with the functor $\text{Def}_{(X,L)}$ where X is a scheme and L is an invertible sheaf. Since a projective Stanley-Reisner scheme X comes equipped with a very ample line bundle I define $\text{Def}_X^a = \text{Def}_{(X,\mathcal{O}_X(1))}$ and it is the versal formal element of this functor I consider.

Let T be an equivelar triangulated torus and X the Stanley-Reisner scheme. Since the equations defining Def_X^a are binomial, we may use the results and techniques of Eisenbud and Sturmfels in [ES96] to realize the smoothing components as toric varieties. The main computational part,

Section 4, is about the cones and lattices that determine these toric varieties (or their normalizations). I then apply these results to statements about the deformations of X in Section 5.

There should be a connection between the results in this paper and moduli of polarized abelian surfaces. In Section 6 I describe a Heisenberg group H_T associated to T . There turns out to be a smooth 3 dimensional subspace $\mathcal{M} \subset \text{Def}_X^a$ containing all isomorphism classes of smoothings of X . Moreover, the fibers are exactly the H_T invariant deformations of X . There is a finite group acting on \mathcal{M} inducing isomorphisms on the fibers and the quotient space $\bar{\mathcal{M}}$ can be easily described in toric geometric terms.

To understand the connection to moduli one should extend the results in this paper to the non-polyhedral equivelar maps on the torus. These should correspond to moduli where the polarization class is not represented by a very ample line bundle. This is at the moment work in progress.

In principle one can write equations for abelian surfaces in \mathbb{P}^{n-1} as perturbations of the Stanley Reisner ideal of T . I include some details about this ideal in Section 3. An application can be found in Section 7.

The last section deals in detail with the vertex minimal triangulation of the torus, sometimes called the Möbius torus, with 7 vertices. Here the toric geometry of Def_X^a is extremely nice and leads to a Calabi-Yau 3-fold with Euler number 6. In this case it is also possible to find all the components of Def_X^a and the generic non-smoothable fibers.

It is convenient to work over the ground field \mathbb{C} . Throughout this paper $G^* := \text{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$ is the character group of G .

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1. PRELIMINARIES

1.1. Equivelar triangulations of the torus. I start by defining the main combinatorial object in this paper. For details and proofs see [BK08]. A map on a surface is called *equivelar* if there are numbers p and q such that every vertex is q -valent and every facet contains exactly p vertices. On a torus we can only have (p, q) equal $(6, 3)$, $(3, 6)$ or $(4, 4)$.

We will consider *triangulated* tori, i.e. the case $p = 3, q = 6$. In this paper we need honest triangulations and assume the map is *polyhedral*. This means that the intersection of two triangles is a common face (i.e. empty, vertex or edge).

Every triangulated torus (also the non-polyhedral) are obtained as a quotient of the regular tessellation of the plane by equilateral triangles ([Neg83]).

Denote this tessellation $\{3, 6\}$. We will need to make this explicit and will refer to the following as the *standard description*.

We may describe $\{3, 6\}$ as an explicit triangulation of \mathbb{R}^2 , i.e. we always assume a chosen origin 0 and coordinates (x, y) . We may assume the vertices of $\{3, 6\}$ form the rank 2 lattice spanned by $(1, 0)$ and $\frac{1}{2}(1, \sqrt{3})$. Denote this lattice by \mathbb{T} . We may think of \mathbb{T} as the translation subgroup of $\text{Aut}(\{3, 6\})$. Now let $\Gamma \subseteq \mathbb{T}$ be a sublattice of finite index and set $T = \{3, 6\}/\Gamma$. Then T is a (not necessarily polyhedral) equivelar triangulated torus. Such a triangulation is called *chiral* if the \mathbb{Z}_6 rotation on $\{3, 6\}$ descends to T . Chiral maps on the torus were studied and classified in [Cox49].

We may assume that Γ is generated by $a(1, 0)$ and $b(1, 0) + c\frac{1}{2}(1, \sqrt{3})$ for integers a, b, c with $ac \neq 0$, i.e. it is the image of

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

in the above basis for \mathbb{T} . In [BK08, Proposition 2] it is shown that two such matrices, M_1 and M_2 , represent isomorphic triangulated tori if and only if $M_2 = PM_1Q$, with $Q \in \text{GL}_2(\mathbb{Z})$ and P in the D_6 subgroup generated by the rotation ρ and reflection σ ,

$$\rho = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \sigma = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

1.2. Deformations of Stanley-Reisner schemes. I refer to [AC09] and the references there for definitions, details and proofs about deformations of Stanley-Reisner schemes. As a general reference for deformation theory see [Ser06].

Given a simplicial complex $\mathcal{K} \subseteq 2^{[n]}$, we may construct the corresponding Stanley-Reisner \mathbb{C} -algebra $A_{\mathcal{K}}$ and the projective Stanley-Reisner scheme $\mathbb{P}(\mathcal{K}) = \text{Proj } A_{\mathcal{K}} \subseteq \mathbb{P}^{n-1}$. Note that $\mathbb{P}(\mathcal{K})$ comes with a very ample line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{K})}(1)$.

The scheme $\mathbb{P}(\mathcal{K})$ looks like the geometric realization of \mathcal{K} . It is a union of irreducible components $X_F = \mathbb{P}^{\dim F}$, F a facet of \mathcal{K} , intersecting as in \mathcal{K} . There is also a natural open affine cover described in terms of Stanley-Reisner rings. Recall that the *link* of a face is

$$\text{lk}(f, \mathcal{K}) = \{g \in \mathcal{K} : g \cap f = \emptyset \text{ and } g \cup f \in \mathcal{K}\}.$$

If $f \in 2^{[n]}$, let $D_+(x_f) \subseteq \mathbb{P}(\mathcal{K})$ be the chart corresponding to homogeneous localization of $A_{\mathcal{K}}$ by the powers of x_f . Then $D_+(x_f)$ is empty unless $f \in \mathcal{K}$ and if $f \in \mathcal{K}$ then

$$D_+(x_f) = \mathbb{A}(\text{lk}(f, \mathcal{K})) \times (\mathbb{C}^*)^{\dim f}$$

where $\text{lk}(f, \mathcal{K}) := \{g \in \mathcal{K} : g \cap f = \emptyset \text{ and } g \cup f \in \mathcal{K}\}$ is the link of f in \mathcal{K} and $\mathbb{A}(-)$ denotes the Spec of the corresponding Stanley-Reisner ring.

The cohomology of the structure sheaf is given by $H^p(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})}) \simeq H^p(\mathcal{K}; \mathbb{C})$ (Hochster). If \mathcal{K} is an *orientable* combinatorial manifold then the

canonical sheaf is trivial ([BE91a, Theorem 6.1]). Thus a smoothing of such a $\mathbb{P}(\mathcal{K})$ would yield smooth schemes with trivial canonical bundle.

In the surface case deformations of Stanley-Reisner schemes may include non-algebraic schemes. It is therefore convenient to work with the functor $\text{Def}_{(X,L)}$ where X is a scheme and L is an invertible sheaf. (See [Ser06, 3.3.3] and [AC09, 3].) We defined $\text{Def}_{\mathbb{P}(\mathcal{K})}^a = \text{Def}_{(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})}(1))}$. If \mathcal{K} is a combinatorial manifold without boundary then

$$\text{Def}_{\mathbb{P}(\mathcal{K})}^a(\mathbb{C}[\epsilon]) \simeq H^0(\mathbb{P}(\mathcal{K}), \mathcal{T}_{\mathbb{P}(\mathcal{K})}^1) \simeq T_{A_{\mathcal{K}},0}^1$$

and $H^0(\mathbb{P}(\mathcal{K}), \mathcal{T}_{\mathbb{P}(\mathcal{K})}^2)$ contains all obstructions for $\text{Def}_{\mathbb{P}(\mathcal{K})}^a$ ([AC09, Theorem 6.1]). For certain surfaces the versal base space for $\text{Def}_{\mathbb{P}(\mathcal{K})}^a$ may be computed and as we shall see this is particularly nice for equivelar triangulated tori.

It follows from the results in [AC09] that if \mathcal{K} is a combinatorial manifold without boundary and all vertices have valency greater than or equal 5, then $T_{A_{\mathcal{K}},0}^1$ is the \mathbb{C} vector space on the edges of \mathcal{K} . Since \mathcal{K} is a manifold, the link of an edge must be two vertices, i.e. $\text{lk}(\{p, q\}) = \{\{i\}, \{j\}\}$. If $\varphi_{p,q} \in T_{A_{\mathcal{K}},0}^1$ is the basis element corresponding to $\{p, q\}$ and x_m is in the Stanley-Reisner ideal, then

$$\varphi_{p,q}(x_m) = \begin{cases} \frac{x_m x_p x_q}{x_i x_j} & \text{if } \{i, j\} \subseteq m \\ 0 & \text{otherwise.} \end{cases}$$

There is a natural $(n-1)$ -dimensional torus action on $\text{Proj}(A_{\mathcal{K}})$ where $[\lambda_0, \dots, \lambda_{n-1}] \in (\mathbb{C}^*)^n / \mathbb{C}^*$ takes $x_i \in A_{\mathcal{K}}$ to $\lambda_i x_i$. Note that the induced action on a $\varphi_{p,q} \in T_{A_{\mathcal{K}},0}^1$ as above is

$$\varphi_{p,q} \mapsto \frac{\lambda_p \lambda_q}{\lambda_i \lambda_j} \varphi_{p,q}.$$

If $t_{p,q}$ is the corresponding coordinate function on the versal base space then the action is the contragredient, i.e. $t_{p,q} \mapsto (\lambda_i \lambda_j / \lambda_p \lambda_q) t_{p,q}$.

1.3. Binomial ideals. In [ES96] Eisenbud and Sturmfels prove, among other things, that every binomial ideal has a primary decomposition all of whose primary components are binomial. I review here some of the results and notions from that paper. (See also [DMM08]).

If $w = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$, write $t^w = \prod t_i^{a_i}$ for a monomial in $P = k[t_1, \dots, t_n]$, k for the time being is any algebraically closed field. A binomial is a polynomial with at most two terms, $at^v - bt^w$ with $a, b \in k$. A binomial ideal is an ideal of P generated by binomials.

For an integer vector v , let v_+ and v_- , both with non-negative coordinates, be the positive and negative part of v , i.e. $v = v_+ - v_-$. In general define, for a sublattice $L \subseteq \mathbb{Z}^n$ the *lattice ideal* of L by

$$I_L = \langle t^{v_+} - t^{v_-} : v \in L \rangle \subseteq k[t_1, \dots, t_n].$$

More generally for any character $\rho \in \text{Hom}_{\mathbb{Z}}(L, k^*)$, define

$$I_{L,\rho} = \langle t^{v_+} - \rho(v) t^{v_-} : v \in L \rangle.$$

If ρ' is an extension of ρ to \mathbb{Z}^n , then the automorphism $t_i \mapsto \rho'(\varepsilon_i)t_i$ induces an isomorphism $I_L \simeq I_{L,\rho}$.

Define the *saturation* of L in \mathbb{Z}^n as the lattice

$$\text{Sat } L = \{v \in \mathbb{Z}^n : dv \in L \text{ for some } d \in \mathbb{Z}\}.$$

Note $\text{Sat } L/L$ is finite. The lattice L is *saturated in \mathbb{Z}^n* if $\text{Sat } L = L$. The lattice ideal is a prime ideal if and only if L is saturated ([ES96, Theorem 2.1]). In fact [ES96, Corollary 2.3] states that

$$I_L = \bigcap_{\rho \in (\text{Sat } L/L)^*} I_{\text{Sat } L,\rho}$$

is a minimal primary decomposition.

Let I be a binomial ideal. Let $\mathcal{Z} \subseteq \{1, \dots, n\}$ and let $\mathfrak{p}_{\mathcal{Z}} = \langle t_i : i \notin \mathcal{Z} \rangle$ and $\mathbb{Z}^{\mathcal{Z}} \subset \mathbb{Z}^n$ the sublattice spanned by the standard basis elements $e_i, i \in \mathcal{Z}$. The result we need from [ES96] is the following. (It is not stated in the following form in that paper and in fact much stronger results are proven there.)

Theorem 1.1. *If characteristic k is 0, the associated primes of the binomial ideal I are all of the form $I_{\text{Sat } L_{\mathcal{Z},\rho}} + \mathfrak{p}_{\mathcal{Z}}$ for some sublattice $L_{\mathcal{Z}} \subseteq \mathbb{Z}^{\mathcal{Z}}$.*

For simplicity let us assume I is generated by pure binomials of the form $t^v - t^w$. Define the *exponent vector* of $t^v - t^w$ to be $v - w \in \mathbb{Z}^m$. Let $L \subset \mathbb{Z}^m$ be the sublattice spanned by the exponent vectors of the generators of I . It follows from the above and [ES96, Theorem 6.1] that the $I_{\text{Sat } L,\rho}$ will be minimal prime ideals for I . It follows from the theorem that all other associated primes must contain some variable t_i .

1.4. Gorenstein, reflexive and Cayley cones. I recall some notions originally introduced in [BB97] in connection with mirror symmetry. A general reference is [BN08]. By cone we will mean a rational finite polyhedral cone. If $M \simeq \mathbb{Z}^n$ then set as usual $N = \text{Hom}(M, \mathbb{Z})$. A cone $\sigma \subseteq M_{\mathbb{R}} = M \otimes \mathbb{R}$ is called *Gorenstein* if there exists $n_{\sigma} \in N$ with $\langle v, n_{\sigma} \rangle = 1$ for all primitive generators $v \in M$ of rays of σ . This means that the affine toric variety X_{σ} is Gorenstein.

The cone σ is called *reflexive* if the dual cone σ^{\vee} is also Gorenstein. Let $m_{\sigma^{\vee}} \in M$ be the determining lattice point. The number $r = \langle m_{\sigma^{\vee}}, n_{\sigma} \rangle$ is the *index* of the reflexive cone σ .

A polytope in $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is called a lattice polytope if its set of vertices is in M . Let $\Delta_1, \dots, \Delta_r \subseteq L_{\mathbb{R}}$ be lattice polytopes in a rank d lattice L . Let $M = L \oplus \mathbb{Z}^r$, where $\{\epsilon_1, \dots, \epsilon_r\}$ is the standard basis for \mathbb{Z}^r . The cone

$$\sigma = \{(\lambda_1, \dots, \lambda_r, \lambda_1 x_1 + \dots + \lambda_r x_r) \in M_{\mathbb{R}} : \lambda_i \in \mathbb{R}_{\geq 0}, x_i \in \Delta_i, i = 1, \dots, r\}$$

is called the *Cayley cone* associated to $\Delta_1, \dots, \Delta_r$. It is a Gorenstein cone with $n_{\sigma} = \epsilon_1^* + \dots + \epsilon_r^*$. A reflexive Gorenstein cone of index r is *completely split* if it is the Cayley cone associated to r lattice polytopes.

1.5. Heisenberg groups. Heisenberg groups are an important ingredient in the construction of moduli spaces for polarized abelian varieties. The following construction is based on [Mum66]. Since we are dealing with surfaces I describe only the 2 dimensional case.

Let $\delta = (d_1, d_2)$ be a list of elementary divisors, i.e. d_i are positive integers and $d_1|d_2$. Set $K(\delta) = \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2}$ with character group $K(\delta)^* = \mu_{d_1} \times \mu_{d_2}$. (Here $\mu_d = \mathbb{Z}_d^*$ are the d 'th roots of unity.) Define the abstract finite *Heisenberg group* H_δ as the extension

$$1 \rightarrow \mu_{d_2} \rightarrow H_\delta \rightarrow K(\delta) \oplus K(\delta)^* \rightarrow 0$$

where multiplication in $\mu_{d_2} \oplus K(\delta) \oplus K(\delta)^*$ is defined by

$$(\omega, \tau, \sigma) \cdot (\omega', \tau', \sigma') = (\omega \cdot \omega' \cdot \sigma'(\tau), \tau + \tau', \sigma \cdot \sigma').$$

If $n = d_1 d_2 = |K(\delta)|$ then H_δ has a unique n -dimensional irreducible representation $V(\delta)$ in which the center $\mu_{d_2} \subset \mathbb{C}^*$ acts by its natural character ([Mum66, Proposition 3]). One may realize $V(\delta)$ as the vector space of \mathbb{C} valued functions f on $K(\delta)$. Then the action is defined by

$$((\omega, \tau, \sigma) \cdot f)(\tau') = \omega \cdot \sigma(\tau') \cdot f(\tau + \tau').$$

The representation $V(\delta)$ is known as the *Schrödinger representation* of the Heisenberg group.

2. OVERVIEW

Let T be an equivelar triangulation of the torus with n vertices and $X \subset \mathbb{P}^{n-1}$ the projective Stanley-Reisner scheme associated to T . Let Γ be the sublattice of \mathbb{T} defining T ; i.e. $T = \{3, 6\}/\Gamma$ and $G = \mathbb{T}/\Gamma$.

There are three for us important elements of \mathbb{T} and I will call them and their images in G the *principal translations*. In the standard description (see Section 1.1) they are

$$\tau_1 = (1, 0) \quad \tau_2 = \frac{1}{2}(-1, \sqrt{3}) \quad \tau_3 = \frac{1}{2}(-1, -\sqrt{3}).$$

There is the relation $\tau_1 + \tau_2 + \tau_3 = 0$. In \mathbb{T} and therefore also in G , any pair of them generate the group.

The following proposition is our central observation and allows us a natural identification of the elements of G and $\text{vert } T$, a fact we will use throughout.

Proposition 2.1. *The edge graph of T is the Cayley graph of G with respect to the principal translations. In particular the action of G on $\text{vert } T$ is simply transitive and the set of edges of T is partitioned by the three G orbits of cardinality n :*

$$\{\{p, \tau_k(p)\} : p \in \text{vert } T\}$$

for $k = 1, 2, 3$.

Proof. The edge graph of $\{3, 6\}$ is clearly the Cayley graph of \mathbb{T} with respect to the principal translations. \square

An edge in T is of *type* k if it is of the form $\{p, \tau_k(p)\}$. This is the natural coloring of the Cayley graph. The link of p in T will be the cycle

$$(\tau_1(p), -\tau_3(p), \tau_2(p), -\tau_1(p), \tau_3(p), -\tau_2(p)).$$

This shows that $\tau_i \neq \pm\tau_j$, for otherwise T would not be polyhedral.

Notation. When describing the interaction between the principal translations it will be useful to have the following convention for the indices. If k is an element in $\{1, 2, 3\}$ then I will use the indices i, j to represent the remaining two elements of $\{1, 2, 3\} \setminus \{k\}$. I will refer to this as the ijk -convention.

Recall from Section 1.2 that the tangent space of Def_X^a has basis $\varphi_{p,q}$, $\{p, q\} \in T$. In our new notation the corresponding perturbation is

$$\varphi_{p,\tau_k(p)}(x_m) = \begin{cases} \frac{x_m x_p x_{\tau_k(p)}}{x_{-\tau_i(p)} x_{-\tau_j(p)}} & \text{if } \{-\tau_i(p), -\tau_j(p)\} \subseteq m \\ 0 & \text{otherwise.} \end{cases}$$

Let $t_{p,q}$ be the dual basis of coordinate functions on \mathbb{C}^{3n} .

For each $p \in \text{vert } T$ construct the matrix

$$(2.1) \quad \begin{bmatrix} t_{p,\tau_1(p)} & t_{p,\tau_2(p)} & t_{p,\tau_3(p)} \\ t_{p,-\tau_1(p)} & t_{p,-\tau_2(p)} & t_{p,-\tau_3(p)} \end{bmatrix}$$

and take the 2×2 minors. This yields $3n$ quadratic binomials. Let I to be the ideal generated by them.

Theorem 2.2 ([AC09] Theorem 6.10). *The ideal I defines a versal base space in $(\mathbb{C}^{3n}, 0)$ for Def_X^a .*

I wish to employ the results on binomial ideals reviewed in Section 1.3. Let $L \subset \mathbb{Z}^{3n}$ be the sublattice spanned by the exponent vectors of the generators of I . Index the standard basis of \mathbb{Z}^{3n} , $\varepsilon_{p,q}$, by the edges of T . The lattice L is spanned by the $3n$ vectors

$$(2.2) \quad \begin{aligned} f_{3,p} &= \varepsilon_{p,\tau_1(p)} + \varepsilon_{p,-\tau_2(p)} - \varepsilon_{p,\tau_2(p)} - \varepsilon_{p,-\tau_1(p)} \\ f_{2,p} &= \varepsilon_{p,\tau_3(p)} + \varepsilon_{p,-\tau_1(p)} - \varepsilon_{p,\tau_1(p)} - \varepsilon_{p,-\tau_3(p)} \\ f_{1,p} &= \varepsilon_{p,\tau_2(p)} + \varepsilon_{p,-\tau_3(p)} - \varepsilon_{p,\tau_3(p)} - \varepsilon_{p,-\tau_2(p)}. \end{aligned}$$

There are relations $f_{1,p} + f_{2,p} + f_{3,p} = 0$ for each p and $\sum_{p \in \text{vert } T} f_{k,p} = 0$ for each k . One checks that indeed these are the generating relations and therefore $\text{rank } L = 2(n-1)$.

The $I_{\text{Sat } L, \rho}$ for $\rho \in (\text{Sat } L/L)^*$ will be minimal prime ideals for I . This means that each $\rho \in (\text{Sat } L/L)^*$ determines a component of the versal base space. I denote these by S_ρ and call them the *main components* of Def_X^a .

Set S to be the component for the trivial ρ , that is S is defined by the toric lattice ideal of $\text{Sat } L$. In the recent literature it is become normal to include non-normal varieties in the term *toric varieties*. I will also do this, thus S is the germ of an affine toric variety. It is in general not normal. Since $\text{rank } \text{Sat } L = 2(n-1)$, $\dim S = n+2$.

There are isomorphisms $\rho' : S \simeq S_\rho$. By their construction, the ρ' are automorphisms of the polynomial ring restricting to the identity on I_L and therefore also on I . Thus $S \simeq S_\rho$ comes from an automorphism of Def_X^a . This implies that the families over the two components are also isomorphic. So from the point of view of deformations it is enough to study S .

As a toric variety the normalization of S , call it \tilde{S} , may be described by a rank $n + 2$ lattice M and a cone $\sigma^\vee \subset M_{\mathbb{R}}$. To find M and σ^\vee we need to find an integral $m \times 3n$ matrix A (for some m) such that $\ker A = \text{Sat } L$. Then set $M = \text{im } A$ and set σ^\vee to be the positive hull of the columns of A in $M_{\mathbb{R}}$. (See e.g. [PT10].) Note it is enough to find A with $\text{rank } A = n + 2$ and $L \subseteq \ker A$, since then $\text{rank } L = \text{rank } \ker A$ and $\ker A$ is obviously saturated, so $\ker A = \text{Sat } L$. Let \mathbb{S} be the subgroup of $\mathbb{Z}_{\geq 0}^m$ generated by the columns of A , thus $S = \text{Spec } \mathbb{C}[\mathbb{S}]$ and $\tilde{S} = \mathbb{C}[\sigma^\vee \cap M]$.

There are two obvious torus actions on Def_X^a and the weights of these actions will give us A . First consider the natural action described in Section 1.2 induced by automorphisms of X . Let $w_{pq} \in \{(a_0, \dots, a_{n-1}) \in \mathbb{Z}^n : \sum a_i = 0\}$ be the weights of this torus action on the basis φ_{pq} . With the ijk -convention

$$w_{p,\tau_k(p)} = e_p + e_{\tau_k(p)} - e_{-\tau_i(p)} - e_{-\tau_j(p)}$$

for $p \in \text{vert } T$, $k = 1, 2, 3$, where e_p are the standard basis for \mathbb{Z}^n .

The coloring of the Cayley graph and the structure of I give us another torus action, not seen on X . Clearly the minors of

$$\begin{bmatrix} \lambda_1 t_{p,\tau_1(p)} & \lambda_2 t_{p,\tau_2(p)} & \lambda_3 t_{p,\tau_3(p)} \\ \lambda_1 t_{p,-\tau_1(p)} & \lambda_2 t_{p,-\tau_2(p)} & \lambda_3 t_{p,-\tau_3(p)} \end{bmatrix}$$

also generate I . Thus the \mathbb{C}^{*3} action, $(\lambda_1, \lambda_2, \lambda_3) \cdot t_{p,\tau_k(p)} = \lambda_k t_{p,\tau_k(p)}$ preserves I .

This leads to the following definition. Write the standard basis for $\mathbb{Z}^3 \oplus \mathbb{Z}^n$ as $\epsilon_1, \epsilon_2, \epsilon_3$ and $e_p, p \in \text{vert } T$. Let A be the $(n+3) \times 3n$ matrix with columns

$$(2.3) \quad A_{p,\tau_k(p)} = \epsilon_k + e_p + e_{\tau_k(p)} - e_{-\tau_i(p)} - e_{-\tau_j(p)}$$

for $p \in \text{vert } T$, $k = 1, 2, 3$. Clearly $L \subseteq \ker A$.

For each $p \in \text{vert } T$, the corresponding row in A has a nice description. In columns indexed by the 6 edges having p as vertex, there is a $+1$. In the 6 columns corresponding to edges in $\text{lk}(p)$ we have -1 . The other entries are 0. Using this one checks that $\text{rank } A = n + 2$.

Let $M = \text{im } A$ and set M' to be the lattice $\mathbb{Z}^3 \oplus \{(a_0, \dots, a_{n-1}) \in \mathbb{Z}^n : \sum a_i = 0\}$, the target of A . For any lattice M let $T_M = M^*$ be the corresponding torus. Consider the finite character group $(M'/M)^*$. By standard toric variety theory, see e.g. [Ful93, 2.2], $T_M = T_{M'}/(M'/M)^*$ and

$$\text{Spec } \mathbb{C}[M \cap \sigma^\vee] = \text{Spec } \mathbb{C}[M' \cap \sigma^\vee]^{(M'/M)^*}.$$

Thus the normalizations of the main components will all be isomorphic to

$$\tilde{S} = (\text{Spec } \mathbb{C}[M \cap \sigma^\vee], 0) = (\text{Spec } \mathbb{C}[M' \cap \sigma^\vee]^{(M'/M)^*}, 0).$$

Our goal is to describe these combinatorial objects to get an as explicit as possible description of the main components.

3. THE STANLEY-REISNER IDEAL OF T

The f -vector of T is $(n, 3n, 2n)$, so one may compute the Hilbert polynomial of A_T , following [Sta96], as

$$h_{A_T}(z) = n \binom{z-1}{0} + 3n \binom{z-1}{1} + 2n \binom{z-1}{2} = nz^2.$$

This agrees with the Hilbert function except in degree 0. In particular one computes that

$$\dim I_2 = \binom{n+1}{2} - 4n = \frac{1}{2}n(n-7).$$

The minimum number of cubic generators on the other hand will depend upon the combinatorics of T .

To compute the number of cubic generators consider first for every edge $\{p, q\}$ the number

$$l_{p,q} = |\text{vert}(\text{lk}(\{p\}, T) \cap \text{lk}(\{q\}, T)) \setminus \text{vert} \text{lk}(\{p, q\}, T)|$$

which can be 0, 1, 2 or 3. By symmetry $l_{p,q}$ will depend only on the type of $\{p, q\}$, so let l_k , $k = 1, 2, 3$, be this common value.

Lemma 3.1. *The minimum number of cubic generators of I_T is $\frac{1}{3}n(l_1 + l_2 + l_3)$.*

Proof. A cubic monomial generator in I_T corresponds to a set $\{p, q, r\}$ of vertices which is a non-face, but for which every subset is an edge. This means exactly that $\{p, q\} \in T$ and $\{r\} \in \text{lk}(\{p\}) \cap \text{lk}(\{q\}) \setminus \text{lk}(\{p, q\})$. In the sum $\sum_{\{p,q\} \in T} l_{p,q}$ we have counted a given such $\{p, q, r\}$ 3 times. \square

The following lemma follows from a simple check. Note that it is not valid for non-polyhedral T .

Lemma 3.2. *With the ijk -convention, l_k is non-zero if and only if $\tau_k = 2\tau_i$ or $\tau_k = 2\tau_j$ or $3\tau_k = 0$.*

Proposition 3.3. *The ideal I_T is generated by quadratic and cubic monomials. The minimum number of quadratic generators is $\frac{1}{2}n(n-7)$.*

Up to isomorphism, the T which need cubic generators for I_T have one of the following standard presentations:

$$\begin{pmatrix} n & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & \frac{n}{3} \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & \frac{n}{3} \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & \frac{n}{3} \end{pmatrix}.$$

If T is presented by $\begin{pmatrix} n & 2 \\ 0 & 1 \end{pmatrix}$, then the minimum

$$\# \text{cubic generators} = \begin{cases} 21 & \text{if } n = 7 \\ 16 & \text{if } n = 8 \\ n & \text{if } n \geq 9. \end{cases}$$

In the three other cases the minimum

$$\# \text{cubic generators} = \begin{cases} 9 & \text{if } n = 9 \\ \frac{1}{3}n & \text{if } n \geq 10. \end{cases}$$

Proof. The only Stanley-Reisner ideal of a 2-dimensional combinatorial manifold that needs quartic generators is the boundary of the tetrahedron.

By Lemma 3.1 and Lemma 3.2 there will no cubic generators unless some $\tau_k = 2\tau_i$ or some $3\tau_k = 0$. In the first case, after a D_6 movement (see Section 1.1) we may assume $\tau_3 = 2\tau_1$. In the standard presentation this means that $(2, 1) \in \Gamma$. One checks using e.g. [BK08, Proposition 3] that, for each n , there is only one isomorphism class with this property and that it is represented by the first matrix in the list. In the second case we may assume $3\tau_1 = 0$ and again check possibilities.

To get the number just count the l_k in each case and use Lemma 3.1. \square

4. ANALYSIS

4.1. The group G and its principal translations. Before proceeding with our analysis of the components I state some facts about $G = \mathbb{T}/\Gamma$ and the τ_k . The proof of the first lemma is an exercise in elementary abelian group theory.

Lemma 4.1. *There are the following relationships involving the principal translations.*

- (i) *The quotient group $G/\langle\tau_k\rangle$ is cyclic and the classes of τ_i and τ_j are both generators. In particular $|\tau_i||\tau_j|/n$ is an integer for all $i \neq j$ in $\{1, 2, 3\}$.*
- (ii) *Let $[g]_i \subseteq G$ be the coset of $\langle\tau_i\rangle$ containing g . For any $g, h \in G$,*

$$|[g]_i \cap [h]_j| = \frac{|\tau_i||\tau_j|}{n}.$$

- (iii) *The number*

$$\gcd\left(\frac{n}{|\tau_i|}, \frac{n}{|\tau_j|}\right) = \frac{n}{\text{lcm}(|\tau_i|, |\tau_j|)}$$

is the same for all $i \neq j$.

I will vary between two presentations of G . First there is the *standard presentation* which is the presentation in Section 1.1. Here G is a quotient

of $\mathbb{T} \simeq \mathbb{Z}^2$ with basis $\tau_1, -\tau_3$ by the image of

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

Then we may take the *symmetric presentation* where we think of G as the quotient of \mathbb{Z}^3 with basis τ_1, τ_2, τ_3 by the image of a matrix

$$R = \begin{pmatrix} 1 & \alpha_{1,1} & \alpha_{2,1} \\ 1 & \alpha_{1,2} & \alpha_{2,2} \\ 1 & \alpha_{1,3} & \alpha_{2,3} \end{pmatrix}.$$

Of course the standard presentation is the symmetric with

$$R = \begin{pmatrix} 1 & a & b \\ 1 & 0 & 0 \\ 1 & 0 & -c \end{pmatrix}.$$

Proposition 4.2. *With the symmetric presentation and the ijk -convention the orders of the principal translations are*

$$|\tau_k| = \frac{n}{\gcd(\alpha_{1,i} - \alpha_{1,j}, \alpha_{2,i} - \alpha_{2,j})}.$$

Proof. The determinant of R is n . Computing it three different ways one sees that $\gcd(\alpha_{1,i} - \alpha_{1,j}, \alpha_{2,i} - \alpha_{2,j})|n$ for all $i \neq j$. Now $|\tau_k|$ is the least positive m with $m\varepsilon_k \in \text{im } R$. Using Cramers rule, this is the least positive m with $m(\alpha_{1,i} - \alpha_{1,j}) \equiv m(\alpha_{2,i} - \alpha_{2,j}) \equiv 0 \pmod{n}$. The result now follows. \square

Here is the the abstract structure of G .

Proposition 4.3. *If*

$$d = \gcd\left(\frac{n}{|\tau_i|}, \frac{n}{|\tau_j|}\right)$$

for $i \neq j$, then the elementary divisors on G are $(d, n/d)$. In particular

$$G \simeq \mathbb{Z}_d \times \mathbb{Z}_{\frac{n}{d}}$$

and G is cyclic if and only if $d = 1$.

Proof. These invariants may be computed from the standard presentation. We have $n = ab$, $n/|\tau_1| = c$, $n/|\tau_2| = \gcd(a, b + c)$ and $n/|\tau_3| = \gcd(a, b)$ (see Proposition 4.2). Thus the d in the statement equals $\gcd(a, b, c)$ as it should. \square

4.2. Sat L and the number of main components. Let B be the $3n \times 3n$ matrix with columns the exponent vectors (2.2) of I . As explained in Section 2, $\text{Sat } L = \ker A$. Thus there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & \mathbb{Z}^{3n} & \longrightarrow & \text{Coker } B \longrightarrow 0 \\ & & \downarrow & & \downarrow = & & \downarrow \\ 0 & \longrightarrow & \text{Sat } L & \longrightarrow & \mathbb{Z}^{3n} & \xrightarrow{A} & M \longrightarrow 0 \end{array}$$

and the Snake Lemma yields an exact sequence

$$0 \rightarrow \text{Sat } L/L \rightarrow \text{Coker } B \xrightarrow{A} M \rightarrow 0.$$

Let d be as in Proposition 4.3.

Proposition 4.4. *There is an isomorphism $\text{Coker } B \simeq F \oplus \mathbb{Z}_d$ where F is free of rank $n + 2$. In particular*

$$\text{Sat } L/L \simeq \mathbb{Z}_d.$$

Proof. We see from the $f_{k,p}$ described in (2.2) that $(\text{Coker } B)^*$ is the set of $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^{*n} \times \mathbb{C}^{*n} \times \mathbb{C}^{*n}$ satisfying

$$(4.1) \quad \frac{\lambda_{1,p}}{\lambda_{1,\tau_1(p)}} = \frac{\lambda_{2,p}}{\lambda_{2,\tau_2(p)}} = \frac{\lambda_{3,p}}{\lambda_{3,\tau_3(p)}} \text{ for all } p \in \text{vert } T.$$

Let $\pi : (\text{Coker } B)^* \rightarrow \mathbb{C}^{*n}$ be the projection on the third factor. Clearly $\ker \pi$ equals

$$\{(\lambda_1, \lambda_2) : \lambda_{1,p} = \lambda_{1,\tau_1(p)}, \lambda_{2,p} = \lambda_{2,\tau_2(p)}, \forall p \in \text{vert } T\} = (\mathbb{C}^*)^{\frac{n}{|\tau_1|}} \times (\mathbb{C}^*)^{\frac{n}{|\tau_2|}}.$$

To prove the statement I will now show that $\text{im } \pi \simeq (\mathbb{C}^*)^{n+2-\frac{n}{|\tau_1|}-\frac{n}{|\tau_2|}} \times (\mathbb{Z}_d)^*$.

Let $\lambda_3 \in \text{im } \pi$. Choose some p and let O_p^1 be the τ_1 orbit of p . From (4.1) we get

$$\prod_{q \in O_p^1} \frac{\lambda_{3,q}}{\lambda_{3,\tau_3(q)}} = \prod_{q \in O_p^1} \frac{\lambda_{1,q}}{\lambda_{1,\tau_1(q)}} = 1.$$

On the other hand, given λ_3 satisfying this relation, choose an arbitrary value for $\lambda_{1,p}$ and set

$$\lambda_{1,r\tau_1(p)} = \lambda_{1,p} \prod_{k=0}^{r-1} \frac{\lambda_{3,(\tau_3+k\tau_1)(p)}}{\lambda_{3,k\tau_1(p)}}$$

to solve (4.1). The same is of course true for τ_2 orbits.

Thus $\text{im } \pi$ is the set of $\lambda \in \mathbb{C}^{*n}$ with

$$(4.2) \quad \prod_{p \in O} \frac{\lambda_p}{\lambda_{\tau_3(p)}} = 1 \text{ for all } \tau_1 \text{ and } \tau_2 \text{ orbits } O.$$

If \mathcal{P}_i are the orbit partitions of $\text{vert } T$ by τ_i , then τ_3 acts transitively on \mathcal{P}_1 and \mathcal{P}_2 (Lemma 4.1). Thus condition (4.2) translates to

$$\prod_{p \in O} \lambda_p = \prod_{q \in O'} \lambda_q \text{ for all } O, O' \in \mathcal{P}_1 \text{ and all } O, O' \in \mathcal{P}_2.$$

For $O \in \mathcal{P}_i$ let this common value be $\mu_i = \prod_{p \in O} \lambda_p$, $i = 1, 2$. There is a homomorphism $\phi : \text{im } \pi \rightarrow \mathbb{C}^{*2}$, $\lambda \mapsto (\mu_1, \mu_2)$. Clearly $\ker \phi$ is the set of $\lambda \in \mathbb{C}^{*n}$ with $\prod_{p \in O} \lambda_p = 1$ for all $O \in \mathcal{P}_1$ and all $O \in \mathcal{P}_2$.

Let R_G be the free abelian group on the elements of G with the regular G action. Set $R_G^{\tau_i}$ to be the invariant sublattice under the action of $\langle \tau_i \rangle$. We may realize $\ker \phi$ as the kernel of the projection $R_G^* \rightarrow (R_G^{\tau_1} + R_G^{\tau_2})^*$. Now

$$\begin{aligned} \text{rank}(R_G^{\tau_1} + R_G^{\tau_2}) &= \text{rank } R_G^{\tau_1} + \text{rank } R_G^{\tau_2} - \text{rank}(R_G^{\tau_1} \cap R_G^{\tau_2}) \\ &= \frac{n}{|\tau_1|} + \frac{n}{|\tau_2|} - 1 \end{aligned}$$

by Lemma 4.1. Thus $\ker \phi \simeq (\mathbb{C}^*)^{n+1 - \frac{n}{|\tau_1|} - \frac{n}{|\tau_2|}}$.

There is one relation between the μ_i namely

$$\prod_{p \in \text{vert } T} \lambda_p = \mu_1^{\frac{n}{|\tau_1|}} = \mu_2^{\frac{n}{|\tau_2|}}.$$

But $d = \gcd(n/|\tau_1|, n/|\tau_2|)$ so $\text{im } \phi \simeq \mathbb{C}^* \times (\mathbb{Z}_d)^*$. Adding this up gives the result. \square

In terms of the structure of Def_X^a one has

Corollary 4.5. *If d is the first elementary divisor of G then the number of main components in Def_X^a is d .*

4.3. The group $(M'/M)^*$. I will compute $(M'/M)^*$. This is of interest in itself, but will also be important for our study of S in Section 6.

From the weight matrix A (2.3) we see that $(M'/M)^* \subset \mathbb{C}^{*3} \times (\mathbb{C}^{*n}/\mathbb{C}^*)$ is defined by

$$(4.3) \quad \mu_k \frac{\lambda_p \lambda_{\tau_k(p)}}{\lambda_{-\tau_i(p)} \lambda_{-\tau_j(p)}} = 1$$

where $(\mu_1, \mu_2, \mu_3) \in \mathbb{C}^{*3}$ and $\lambda \in \mathbb{C}^{*n}/\mathbb{C}^*$ has coordinates indexed by $\text{vert } T$. I will first solve these equations for $\tau_k \in \mathbb{T}$ and $p \in \text{vert}\{3, 6\}$ and then see what happens in the quotient. For this purpose we need the following quadratic parabolic function.

Definition 4.6. Define $q : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ by

$$q(x, y) = \frac{1}{2}(x^2 + y^2 - 2xy - x - y) = \frac{1}{2}((x - y)^2 - (x + y)).$$

Lemma 4.7. *For q there are the equalities*

$$q(mx, my) = mq(x, y) + \frac{1}{2}m(m - 1)(x - y)^2$$

and

$$q(x + z, y + w) = q(x, y) + q(z, w) + (x - y)(z - w).$$

Lemma 4.8. *Choose an origin 0 in $\text{vert}\{3, 6\}$ and $\lambda_0, \lambda_k, \mu_k \in \mathbb{C}^*$, $k = 1, 2, 3$ satisfying*

$$(4.4) \quad \lambda_0^3 = \frac{\lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3}.$$

After these choices, a solution for (4.3) with $\tau_k \in \mathbb{T}$ and $p \in \text{vert}\{3, 6\}$ is unique and given for $p = \alpha\tau_1(0) + \beta\tau_2(0) + \gamma\tau_3(0)$ by

$$(4.5) \quad \lambda_p = \lambda_0^{1-\alpha-\beta-\gamma} \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma \mu_1^{q(\beta,\gamma)} \mu_2^{q(\alpha,\gamma)} \mu_3^{q(\alpha,\beta)}.$$

Proof. We first check that the (4.5) is well defined. Assume $p = \alpha\tau_1(0) + \beta\tau_2(0) + \gamma\tau_3(0) = a\tau_1(0) + b\tau_2(0) + c\tau_3(0)$, then $\alpha - a = \beta - b = \gamma - c = \delta$ for some δ . Now $q(x + \delta, y + \delta) = q(x, y) - \delta$ so the right hand side in (4.5) is changed by multiplication with

$$\left(\frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_0^3 \mu_1 \mu_2 \mu_3} \right)^\delta$$

which equals 1 by condition (4.4).

For the uniqueness consider a vertex p and its link with vertices q_1, \dots, q_6 . One checks directly that if the μ_k , λ_p and 3 of the λ_{q_i} are given, then the relations (4.3) determine the other 3 λ_{q_i} . Now beginning in the origin and working outward we see that all λ_p are determined by λ_0 and $\lambda_{\tau_k(0)}$, $k = 1, 2, 3$.

Finally we must check that these λ_p actually are solutions. We do this only for $k = 1$. After plugging (4.5) into (4.3) and some obvious cancellations we arrive at

$$\frac{\lambda_1 \lambda_2 \lambda_3 \mu_1^{2q(\beta,\gamma)+1} \mu_2^{q(\alpha+1,\gamma)} \mu_3^{q(\alpha+1,\beta)}}{\lambda_0^3 \mu_1^{q(\beta-1,\gamma)} \mu_1^{q(\beta,\gamma-1)} \mu_2^{q(\alpha,\gamma-1)} \mu_3^{q(\alpha,\beta-1)}}.$$

Now $q(x + 1, y) = q(x, y) + x - y$ and $q(x - 1, y) = q(x, y) + y - x + 1$, so this reduces further to

$$\frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_0^3 \mu_1 \mu_2 \mu_3} = 1.$$

□

Now take the quotient by Γ . With the notation of the symmetric presentation let $\alpha_{t,1}\tau_1 + \alpha_{t,2}\tau_2 + \alpha_{t,1}\tau_3$, $t = 1, 2$, be generators of Γ and set $r_t = \alpha_{t,1}\tau_1(0) + \alpha_{t,2}\tau_2(0) + \alpha_{t,1}\tau_3(0)$.

Lemma 4.9. *The character group $(M'/M)^* \subset \mathbb{C}^{*3} \times (\mathbb{C}^{*n}/\mathbb{C}^*)$ consists of the solutions (4.5) under the condition (4.4) with*

$$\lambda_0 = \lambda_{r_1} = \lambda_{r_2} = 1$$

and

$$\mu_1^{A(\alpha_{t,2}-\alpha_{t,3})} \mu_2^{B(\alpha_{t,3}-\alpha_{t,1})} \mu_3^{C(\alpha_{t,1}-\alpha_{t,2})} = 1$$

for $t = 1, 2$ and all integers A, B, C with $A + B + C = 0$.

Proof. Setting $\lambda_0 = 1$ corresponds to the second factor being $\mathbb{C}^{*n}/\mathbb{C}^*$. The first condition is clearly necessary.

For the solution to be valid modulo Γ we must have $\lambda_{m_1 r_1 + m_2 r_2 + p} = \lambda_p$ for all integers m_i and $p \in \text{vert}\{3, 6\}$. Consider first

$$\begin{aligned} \lambda_{m_1 r_1 + m_2 r_2} &= \lambda_{r_1}^{m_1} \lambda_{r_2}^{m_2} \\ &\cdot \left(\mu_1^{(\alpha_{1,2} - \alpha_{1,3})(\alpha_{2,2} - \alpha_{2,3})} \mu_2^{(\alpha_{1,3} - \alpha_{1,1})(\alpha_{2,3} - \alpha_{2,1})} \mu_3^{(\alpha_{1,1} - \alpha_{1,2})(\alpha_{2,1} - \alpha_{2,2})} \right)^{m_1 m_2} \\ &\quad \cdot \prod_{t=1}^2 \left(\mu_1^{(\alpha_{t,2} - \alpha_{t,3})^2} \mu_2^{(\alpha_{t,3} - \alpha_{t,1})^2} \mu_3^{(\alpha_{t,1} - \alpha_{t,2})^2} \right)^{\frac{1}{2} m_t (m_t - 1)} \end{aligned}$$

by Lemma 4.7. The two conditions in the statement imply that this expression equals 1.

Now, in general, if $r = \alpha\tau_1(0) + \beta\tau_2(0) + \gamma\tau_3(0)$ and $p = a\tau_1(0) + b\tau_2(0) + c\tau_3(0)$ then

$$(4.6) \quad \lambda_{r+p} = \lambda_r \lambda_p \mu_1^{(\beta-\gamma)(b-c)} \mu_2^{(\gamma-\alpha)(c-a)} \mu_3^{(\alpha-\beta)(a-b)}$$

by Lemma 4.7.

Let $r = m_1 r_1 + m_2 r_2$ and $A = b - c, B = c - a, C = a - b$ in the (4.6). Set $\beta_{t,ij} = \alpha_{t,i} - \alpha_{t,j}$ to shorten notation. Then the factor involving the μ_i in (4.6) becomes

$$\mu_1^{A(m_1\beta_{1,2,3} + m_2\beta_{2,2,3})} \mu_2^{B(m_1\beta_{1,3,1} + m_2\beta_{2,3,1})} \mu_3^{C(m_1\beta_{1,1,2} + m_2\beta_{2,1,2})} = 1$$

by the second condition. Thus for all p , $\lambda_{m_1 r_1 + m_2 r_2 + p} = \lambda_p$. Choosing $m_1 = 1, m_2 = 0$ and vice versa gives the necessity of the second condition. \square

Proposition 4.10. *There is an extension*

$$1 \rightarrow G^* \rightarrow (M'/M)^* \rightarrow G^* \times (\mathbb{Z}_d)^* \rightarrow 1.$$

In particular $|M'/M| = n^2 d$.

Proof. Consider the projection on the first factor of $\mathbb{C}^{*3} \times \mathbb{C}^{*n-1}$ restricted to $(M'/M)^*$. I claim the kernel is G^* . Indeed, if we set $\mu_i = 1$ in the conditions of Lemma 4.9 we are left with

$$\lambda_1 \lambda_2 \lambda_3 = \lambda_1^{\alpha_{1,1}} \lambda_2^{\alpha_{1,2}} \lambda_3^{\alpha_{1,3}} = \lambda_1^{\alpha_{2,1}} \lambda_2^{\alpha_{2,2}} \lambda_3^{\alpha_{2,3}} = 1.$$

Thus to prove the statement we must show that the image of the projection is the character group of $\mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_{n/d}$.

Consider the relations among the μ_i described in Lemma 4.9. There are 4 generating relations corresponding to $(A, B, C) = (1, -1, 0)$ and $(0, -1, 1)$. We may use the standard presentation of G to compute them. They are

$$\mu_2^a = \mu_2^a \mu_3^a = \mu_1^c \mu_2^{b+c} = \mu_2^{b+c} \mu_3^b = 1.$$

It is an exercise using the algorithm for the Smith normal form to compute that the elementary divisors on the cokernel of

$$(4.7) \quad \begin{pmatrix} 0 & 0 & c & 0 \\ a & 0 & b+c & b+c \\ 0 & a & 0 & b \end{pmatrix}$$

are $(d, d, n/d)$. \square

4.4. The cone σ^\vee . Let $N' \subseteq N$ be the dual lattices of $M \subseteq M'$ and σ the dual cone of σ^\vee . Recall that σ^\vee is the positive hull of the columns of A in $M_{\mathbb{R}}$ and that the columns of A are

$$A_{p, \tau_k(p)} = \epsilon_k + e_p + e_{\tau_k(p)} - e_{-\tau_i(p)} - e_{-\tau_j(p)}$$

for $k = 1, 2, 3$ and $p \in \text{vert } T$. We will need the easily checked lemma.

Lemma 4.11. *If $i \neq j$ and O is a τ_i orbit in $\text{vert } T$, then $\sum_{q \in O} A_{q, \tau_j(q)} = |\tau_i| \epsilon_j$.*

The matrix A has the nice property that the columns generate the rays of σ^\vee .

Lemma 4.12. *Each column of A is a primitive generator in M for a ray of σ^\vee , thus σ^\vee has $3n$ rays.*

Proof. For each edge $\{p, \tau_k(p)\}$ of T let $u_{p, \tau_k(p)} \in N'$ be $2(\epsilon_1^* + \epsilon_2^* + \epsilon_2^*) - (e_p^* + e_{\tau_k(p)}^*)$. If $A_{q, \tau_l(q)}$ is a column of A , then $0 \leq \langle A_{q, \tau_l(q)}, u_{p, \tau_k(p)} \rangle \leq 4$ and equals 0 if and only if $p = q$ and $k = l$. Thus $u_{p, \tau_k(p)} \in \sigma$ and it defines the 1 dimensional face spanned by the column $A_{p, \tau_k(p)}$. \square

Proposition 4.13. *The cone $\sigma^\vee \subseteq M_{\mathbb{R}}$ is a Gorenstein cone. It is a Cayley cone associated to 3 lattice polytopes.*

Proof. If $n_{\sigma^\vee} = \epsilon_1^* + \epsilon_2^* + \epsilon_2^*$, then clearly $\langle n_{\sigma^\vee}, A_{q, \tau_l(q)} \rangle = 1$ so σ^\vee is Gorenstein by Lemma 4.12. It is a Cayley cone by [BN08, Proposition 2.3]. \square

In fact σ^\vee has a finer Cayley structure. First partition each of the sets of type k columns $\{A_{p, \tau_k(p)} : p \in \text{vert } T\}$ into its τ_k orbits. This partitions the set of all $3n$ columns into r cells where

$$r = \frac{n}{|\tau_1|} + \frac{n}{|\tau_2|} + \frac{n}{|\tau_3|}.$$

Index these cells o_1, \dots, o_r and view \mathbb{Z}^r as the free abelian group on the o_i . Let β_i be the standard basis element of \mathbb{R}^r corresponding to o_i . The orbit o_i is of type k if it is a τ_k orbit of type k columns.

Now define the vectors

$$m_p = \sum_{k=1}^3 (e_{\tau_k(p)} - e_{-\tau_k(p)}), \quad p \in \text{vert } T.$$

Since

$$\begin{aligned} A_{-\tau_k(p), p} &= \epsilon_k + e_p + e_{-\tau_k(p)} - e_{(-\tau_i - \tau_k)(p)} - e_{(-\tau_j - \tau_k)(p)} \\ &= \epsilon_k + e_p + e_{-\tau_k(p)} - e_{\tau_j(p)} - e_{\tau_i(p)} \end{aligned}$$

we have $m_p = A_{p, \tau_k(p)} - A_{-\tau_k(p), p}$ for all $k = 1, 2, 3$. Thus $m_p \in M$, so define $M'' \subset M$ to be the sublattice spanned by the m_p .

Let $\tilde{\Delta}$ be the *support* of the Gorenstein cone σ^\vee , i.e. the polytope $\{x \in \sigma^\vee : \langle n_{\sigma^\vee}, x \rangle = 1\}$.

Theorem 4.14. *There is an exact sequence*

$$0 \rightarrow M'' \rightarrow M \rightarrow \mathbb{Z}^r \rightarrow 0$$

where the last map takes $A_{p,\tau_k(p)} \mapsto \beta_i$ if $A_{p,\tau_k(p)} \in o_i$. This projection maps $\tilde{\Delta}$ surjectively on the convex hull of $\{\beta_1, \dots, \beta_r\}$ and therefore determines a Cayley structure of length r on σ^\vee .

Proof. We must show that the application $A_{p,\tau_k(p)} \mapsto \beta_i$ gives us a well-defined morphism $M \rightarrow \mathbb{Z}^r$. This would follow from the following claim:

$$\sum_{k=1}^3 \sum_{p \in \text{vert } T} \alpha_{k,p} A_{p,\tau_k(p)} = 0 \implies \sum_{A_{p,\tau_k(p)} \in o_i} \alpha_{k,p} = 0, \quad i = 1, \dots, r.$$

Assume $\sum \sum \alpha_{k,p} A_{p,\tau_k(p)} = 0$ and that o_i is of type k . We have

$$\begin{aligned} 0 &= \sum_{m=0}^{|\tau_k|-1} m \tau_k \left(\sum_{l=1}^3 \sum_{p \in \text{vert } T} \alpha_{l,p} A_{p,\tau_l(p)} \right) \\ &= \sum_{l=1}^3 \sum_{p \in \text{vert } T} \alpha_{l,p} \sum_{m=0}^{|\tau_k|-1} A_{m\tau_k(p), (m\tau_k + \tau_l)(p)} \\ &= \left(\sum \alpha_{i,p} \right) |\tau_k| \epsilon_i + \left(\sum \alpha_{j,p} \right) |\tau_k| \epsilon_j + \sum \alpha_{k,p} \sum_{m=0}^{|\tau_k|-1} A_{m\tau_k(p), (m+1)\tau_k(p)} \end{aligned}$$

by Lemma 4.11. The right hand term cannot cancel the ϵ_i or ϵ_j term, so must also vanish. Reindex the orbits of type k so they are $o_1, \dots, o_{n/|\tau_k|}$. Now

$$\sum_p \alpha_{k,p} \sum_{m=0}^{|\tau_k|-1} A_{m\tau_k(p), (m+1)\tau_k(p)} = \sum_{i=1}^{n/|\tau_k|} \left(\sum_{A_{p,\tau_k(p)} \in o_i} \alpha_{k,p} \right) \left(\sum_{A_{p,\tau_k(p)} \in o_i} A_{p,\tau_k(p)} \right)$$

so we must show that the $\{ \sum_{A_{p,\tau_k(p)} \in o_i} A_{p,\tau_k(p)} : i = 1, \dots, n/|\tau_k| \}$ is linearly independent.

Let G_k be the subgroup of G generated by τ_k acting on \mathbb{Z}^{n+1} , with basis ϵ_k and $e_p, p \in \text{vert } T$, with $\tau_k(\epsilon_k) = \epsilon_k$ and $\tau_k(e_p) = e_{\tau_k(p)}$. Let $[p]$ denote the G_k orbit of p in $\text{vert } T$. The invariant sublattice $(\mathbb{Z}^{n+1})^{G_k}$ has rank $n/|\tau_k|$ and is spanned by ϵ_k and $\beta_{[p]} = \sum_{q \in [p]} e_q$. (If $n = |\tau_k|$ then of course $\beta_{[p]} = 0$ and $(\mathbb{Z}^{n+1})^{G_k}$ is spanned by ϵ_k .)

Each $\sum_{A_{p,\tau_k(p)} \in o_i} A_{p,\tau_k(p)} \in (\mathbb{Z}^{n+1})^{G_k}$. If $A_{p,\tau_k(p)} \in o_i$ then one computes

$$(4.8) \quad \sum_{A_{q,\tau_k(q)} \in o_i} A_{q,\tau_k(q)} = |\tau_k| \epsilon_k + 2\beta_{[p]} - \beta_{[-\tau_i(p)]} - \beta_{[-\tau_j(p)]}.$$

Now both G_i and G_j act transitively on the set of G_k orbits of $\text{vert } T$ by $\tau_i([p]) = [\tau_i(p)]$ and similarly for G_j (see Lemma 4.1). So, after choosing some $p_0 \in \text{vert } T$ and setting $\bar{\tau}_i$ to be the class of τ_i in G/G_k , index the basis

by $\beta_{[p]} = \beta_m$ if $[p] = m\bar{\tau}_i([p_0])$. Moreover $[-\tau_j(p)] = [(\tau_k + \tau_i)(p)] = [\tau_i(p)]$. Thus, with new indices, the vectors in (4.8) become

$$|\tau_k|\epsilon_k - \beta_{m-1} + 2\beta_m - \beta_{m+1}, \quad m = 0, \dots, \frac{n}{|\tau_k|} - 1$$

(indexed cyclicly) and this is a linearly independent set.

Since $m_p = A_{p, \tau_k(p)} - A_{-\tau_k(p), p}$, for all k , they generate the kernel of $M \rightarrow \mathbb{Z}^r$. The statement about convex hulls follows from the description of the map. The statement about Cayley structures is again [BN08, Proposition 2.3]. \square

Remark. In [BN08] we are told how to find the r polytopes making up the Cayley structure. The support $\tilde{\Delta}$ is the convex hull of the columns of A . Choose some element in each o_i and call it E_i and a basis E_{r+1}, \dots, E_{n+2} for M'' . Thus $\{E_1, \dots, E_{n+2}\}$ is a basis for M . Let E_i^* be the dual basis and set for $i = 1, \dots, r$

$$\tilde{\Delta}_i = \{x \in \tilde{\Delta} : \langle x, E_j^* \rangle = 0 \text{ for } j \in \{1, \dots, r\} \setminus \{i\}\}.$$

Write $\tilde{\Delta}_i = \Delta_i \times E_i$ where Δ_i is a lattice polytope in $M''_{\mathbb{R}}$. The cone σ^\vee is the Cayley cone associated to $\Delta_1, \dots, \Delta_r$.

5. DEFORMATIONS

We may pull back the family over S to the normalization \tilde{S} , which is finite and generically injective over S . Thus if we are only interested in which fibers occur, then we may as well work on \tilde{S} .

Let R be the local ring of Def_X^a . In the proof of Theorem 2.2 in [AC09] we constructed a *local formal model* of the versal family over Def_X^a . That is a collection \mathcal{U}_p , $p \in \text{vert } T$, of affine schemes and deformations $\mathcal{U}_p \rightarrow \text{Def}_X^a$ of U_p such that over $R_n = R/\mathfrak{m}^{n+1}$, the $\mathcal{U}_p \times_{\text{Def}_X^a} \text{Spec } R_n$ could be glued to form a formal versal deformation $X_n \rightarrow \text{Spec } \hat{R}_n$. Thus if $\mathcal{X} \rightarrow \text{Def}_X^a$ is a formally versal deformation, then

$$(\mathcal{X} \times_{\text{Def}_X^a} \text{Spec } R_n)|_{U_p} \simeq \mathcal{U}_p \times_{\text{Def}_X^a} \text{Spec } R_n$$

as formal deformations of U_p .

We may therefore apply the following application of Artin approximation.

Theorem 5.1. *Let R be a local k -algebra and assume $X \rightarrow \text{Spec } R$ and $Y \rightarrow \text{Spec } R$ are two deformations of X_0 with isomorphic associated formal deformations. Then $X \setminus X_0$ is smooth near X_0 if and only if $Y \setminus X_0$ is smooth near X_0 .*

Proof. Let $x \in X_0$. By assumption $\hat{\mathcal{O}}_{X,x} \simeq \hat{\mathcal{O}}_{Y,x}$, thus by the variant of Artin approximation theorem in [Art69, Corollary 2.6], X and Y are locally isomorphic for the étale topology near x . \square

We know the \mathcal{U}_p in detail - see [AC09, Proof of 6.10]. Label the coordinates of \mathbb{P}^{n-1} by x_p , $p \in \text{vert } T$. On U_p denote the 6 coordinates by $y_{p,\pm k} =$

$x_{\pm\tau_k(p)}/x_p$, $k = 1, 2, 3$. Then \mathcal{U}_p is defined by the ideal generated by the 9 equations

$$(5.1) \quad \begin{aligned} y_{p,\mp i}y_{p,\mp j} + t_{p,\pm\tau_k(p)}y_{p,\pm k} & \quad k = 1, 2, 3 \\ y_{p,k}y_{p,-k} - t_{p,-\tau_i(p)}t_{p,\tau_j(p)} & \quad k = 1, 2, 3. \end{aligned}$$

Recall that $t_{p,-\tau_i(p)}t_{p,\tau_j(p)} = t_{p,\tau_i(p)}t_{p,-\tau_j(p)}$ in Def_X^a so the last equation makes sense.

Note that if the coordinates of Def_X^a corresponding to edges of the same type are equated, $t_{p,\tau_k(p)} = t_{q,\tau_k(q)}$ for all $p, q \in \text{vert } T$, the minors of the matrices (2.1) vanish. This defines a smooth 3-dimensional subspace \mathcal{M} of Def_X^a . Recall from Section 1.2 that the action of G on Def_X^a is the same as the action on the edges of T , i.e. $g \cdot t_{pq} = t_{g(p),g(q)}$. It follows immediately that $\mathcal{M} = (\text{Def}_X^a)^G$.

In toric terms we may describe \mathcal{M} this way. Consider the projection on the first factor $p_1 : M' \rightarrow \mathbb{Z}^3$. The restriction to M is surjective and the induced map $M_{\mathbb{R}} \rightarrow \mathbb{R}^3$ maps σ^\vee and \mathbb{S} onto the positive octant. Thus we have a closed embedding of $\mathbb{C}^3 = \text{Spec } \mathbb{C}[\mathbb{Z}_{\geq 0}^3]$ into both $\text{Spec } \mathbb{C}[M \cap \sigma^\vee]$ and $\text{Spec } \mathbb{C}[\mathbb{S}]$. It follows that \mathcal{M} lies in the toric component S . The inclusion $\mathcal{M} \subset S$ is the surjection $\mathbb{C}[\mathbb{S}] \rightarrow \mathbb{C}[\mathbb{Z}_{\geq 0}^3]$ induced by the projection $p_1 : M \rightarrow \mathbb{Z}^3$.

Let K be the image of M under the natural projection

$$M' \rightarrow \{(a_0, \dots, a_{n-1}) \in \mathbb{Z}^n : \sum a_i = 0\}.$$

This yields an inclusion $T_K \subseteq T_M$ which corresponds to the natural $(\mathbb{C}^*)^n/\mathbb{C}^*$ action on Def_X^a induced by the action on X (see Section 1.2). Now T_M and therefore T_K are subspaces of both S and \tilde{S} . We will need the following easily proven lemma.

Lemma 5.2. *Every point in T_M is in a T_K orbit of a point in $T_M \cap \mathcal{M}$.*

In terms of deformations

Lemma 5.3. *The fibers over a T_K orbit in S (or \tilde{S}) are isomorphic.*

Proof. This probably follows from general principles since the action of T_K is induced by \exp of the Lie algebra action of $H^0(X, \Theta_X)$ on T_X^1 . One sees this directly though by noting that T_K acts as automorphisms on \mathcal{U}_p compatible with the formal gluing (see [AC09, Proof of 6.10]). \square

Theorem 5.4. *The main components are the only smoothing components of Def_X^a and the discriminant of \tilde{S} is $\tilde{S} \setminus T_M$.*

Proof. By Theorem 1.1, on a non-main component some $t_{p,\tau_k(p)} = 0$. After looking at the equations (5.1) of the local formal model we conclude that \mathcal{U}_p will be singular, in fact reducible. But then by Theorem 5.1, \mathcal{X} cannot contain a smooth fiber over this component. By standard toric geometry the same argument applies to fibers over $\tilde{S} \setminus T_M$.

It remains to show that the fibers over T_M are smooth. Consider a one parameter sub-family over $C \subset \mathcal{M} \subseteq S$. We may assume C is given by $t_{p,\tau_1(p)} = at, t_{p,\tau_2(p)} = bt, t_{p,\tau_3(p)} = ct$ for some $a, b, c \in \mathbb{C}$. Plug this into the equations (5.1) and one sees that if $abc \neq 0$ the charts over this curve have an isolated singularity at 0. Thus if $abc \neq 0$, Theorem 5.1 and generic smoothness imply that \mathcal{X}_C is a smoothing. Since it is a one-parameter smoothing the nearby fibers will all be smooth. Thus $T_M \cap \mathcal{M}$ has only smooth fibers, but then by Lemma 5.2 and Lemma 5.3 the same is true for T_M . \square

6. MODULI

I will construct the Heisenberg group $H_{(d,n/d)}$ from G . After choosing an origin in $\{3, 6\}$ there is a one to one correspondence $G \rightarrow \text{vert } T$ given by $\tau \mapsto \tau(0)$. As before label the coordinates of \mathbb{P}^{n-1} by $x_p, p \in \text{vert } T$.

The group $G \subseteq \text{Aut } T$ acts on the coordinate functions by $\tau(x_p) = x_{-\tau(p)}$ and G^* acts by $\sigma(x_p) = \sigma(\tau_p)^{-1}x_p$. Taken together this defines an inclusion $G \oplus G^* \hookrightarrow \text{PGL}_n(\mathbb{C})$. Now construct a Heisenberg group $H_T \simeq H_{(d,n/d)}$ with Schrödinger representation as in Section 1.5.

Lemma 6.1. *A point in Def_X^a is H_T invariant if and only if it is G invariant.*

Proof. The induced action of G^* on Def_X^a is trivial. In fact the proof of Proposition 4.10 shows that G^* is the kernel of $(\mathbb{C}^*)^{n-1} \rightarrow \text{GL}(\text{Def}_X^a(\mathbb{C}[\epsilon]))$. \square

Consider the $\mathbb{Z}^3 \subseteq M'$ spanned by ϵ_1, ϵ_2 and ϵ_3 and set $\bar{M} = \mathbb{Z}^3 \cap M$. There is an exact sequence $0 \rightarrow \bar{M} \rightarrow M \rightarrow K \rightarrow 0$. The intersection $\sigma^\vee \cap \bar{M}_{\mathbb{R}}$ is the positive octant $\mathbb{R}_{\geq 0}^3$ since

$$\epsilon_k = \frac{1}{n} \sum_{p \in \text{vert } T} A_{p,\tau_k(p)} \in \sigma^\vee.$$

Let $\bar{M} = \text{Spec } \mathbb{C}[M \cap \mathbb{R}_{\geq 0}^3]$ be the corresponding 3-dimensional toric variety. If $\bar{G} = \mathbb{Z}^3/\bar{M}$ then $\bar{M} = \mathbb{C}^3/\bar{G}^*$. We have already seen \bar{G} in Proposition 4.10 and know that as abstract group it is isomorphic to $G \times \mathbb{Z}_d$.

I state the following lemma for lack of reference, the proof is straightforward.

Lemma 6.2. *Let the cone σ and the lattice N determine the affine toric variety U_σ . Assume the composition of lattice maps $N' \hookrightarrow N \twoheadrightarrow N''$ is injective, induces an isomorphism $\sigma' = \sigma \cap N'_{\mathbb{R}} \simeq \sigma'' = \text{im } \sigma \subset N''_{\mathbb{R}}$ and $\text{rank } N' = \text{rank } N''$. If $K = \ker[N \twoheadrightarrow N'']$, then $\ker[T_{N'} \rightarrow T_{N''}] = T_{N'} \cap T_K$ and it is the stabilizer of $U_{\sigma'} \subset U_\sigma$ in T_K .*

Theorem 6.3. *The composition*

$$\mathbb{C}[M \cap \mathbb{R}_{\geq 0}^3] \hookrightarrow \mathbb{C}[M \cap \sigma^\vee] \twoheadrightarrow \mathbb{C}[\mathbb{Z}_{\geq 0}^3]$$

where the last map is induced by $p_1 : M \rightarrow \mathbb{Z}^3$, is injective and realizes $\bar{\mathcal{M}}$ as \mathcal{M}/\bar{G}^* . This identity associates an isomorphism class of H_T invariant abelian surfaces to each point of $\bar{\mathcal{M}}$.

Proof. The injectivity follows from the injectivity of the composition $\bar{M} \subseteq M \rightarrow \mathbb{Z}^3$. Dualizing this composition we arrive in the situation of Lemma 6.2 with N' dual to \mathbb{Z}^3 and N'' dual to \bar{M} . From toric geometry it follows that $\bar{G}^* \simeq \ker[T_{N'} \rightarrow T_{N''}]$. Thus by Lemma 6.2, \bar{G}^* is isomorphic to the stabilizer subgroup of \mathcal{M} in T_K . The result now follows from the identification $\mathcal{M} = (\text{Def}_X^a)^G$, Lemma 5.3 and Lemma 6.1. \square

Remark. Note that $\bar{\mathcal{M}}$ is the normalization of $\text{Spec } \mathbb{C}[\mathbb{S} \cap \mathbb{Z}^3]$. Theorem 6.3 seems to imply that we see the moduli space for abelian surfaces with level-structure of type $(n, n/d)$ as an open subset of one of these spaces. We may at least think of them as representing the germ at a “deepest” boundary point. This type of claim presupposes an analysis of degenerate abelian surfaces arising from the non-polyhedral equivelar maps on the torus, which is at the moment work in progress.

With the standard description \bar{G} is the cokernel of the matrix (4.7) in the proof of Proposition 4.10. In each particular case it is straightforward to describe the action of \bar{G} and thus the singularities of $\bar{\mathcal{M}}$. Here are two examples.

Example 6.4. Consider the case where G is cyclic and one of the τ_i generate G . We may assume that this is τ_1 , so the standard presentation is given by

$$\begin{pmatrix} n & b \\ 0 & 1 \end{pmatrix}$$

where $2 \leq b \leq n - 2$. (Not all of these are polyhedral.) One computes that the dual lattice,

$$\bar{N} = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{n}(b(b+1), -b, b+1)$$

so the action of \bar{G} is generated by $\text{diag}(\zeta_n^{b(b+1)}, \zeta_n^{-b}, \zeta_n^{b+1})$ where ζ_n is a primitive n 'th root of unity. This will yield an *isolated* quotient singularity if and only if n is coprime to both b and $b+1$. This is true if and only if all three τ_k generate G .

The quotient singularity will be *Gorenstein* if and only if $1 + b + b^2 \equiv 0 \pmod{n}$, which again is equivalent to T being chiral. Indeed, if

$$\rho = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

generates the 6-fold rotational symmetry in the standard description, then T is chiral if and only if Γ , as translation subgroup of $\text{Aut}(\{3, 6\})$, is fixed by conjugacy with ρ . This is again if and only if $\rho(\Gamma) \subset \Gamma$, when we now

view Γ as a sublattice of \mathbb{Z}^2 . The latter is equivalent to

$$\begin{pmatrix} n & b \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n & b \\ 0 & 1 \end{pmatrix}$$

being an integral matrix. The condition for this is exactly $1 + b + b^2 \equiv 0 \pmod n$.

Example 6.5. Consider next the case where G is presented by

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

This is polyhedral if $a, c \geq 3$. In this case $d = \gcd(a, c)$ and one easily computes that \bar{M} is the image of

$$\begin{pmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Thus $\bar{\mathcal{M}} = \text{Spec } \mathbb{C}[x^c, y^d, z^a] \simeq \mathbb{C}^3$.

7. THE VERTEX MINIMAL TRIANGULATION T_7

From the Euler formula $v - e + f = 0$ and the fact that $3f = 2e$ for surface manifolds, one concludes that a triangulated torus must have at least 7 vertices. There is exactly one such triangulation and it is equivelar. It is sometimes called the Möbius torus, since he gave the first description in 1861. In 1949 Császár gave the first polyhedral realization of the triangulation in 3-space. See e.g. [BE91b] and the references therein.

Call this triangulation T_7 . The group G is \mathbb{Z}_7 and the standard presentation is given by

$$\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}$$

so the pair of divisors is $(1, 7)$. It is chiral and the automorphism group is the Frobenius group $F_{42} = \mathbb{Z}_7 \rtimes \mathbb{Z}_6$. The relations among the τ_k are $\tau_2 = 4\tau_1$ and $\tau_3 = 2\tau_1$.

It is the only polyhedral triangulation with complete edge graph, i.e. the edge graph is the complete graph K_7 . The Stanley-Reisner ideal is generated by 21 cubic monomials (see Section 3).

Since $d = 1$, Def_X^a has one main component, the unique smoothing component S .

Theorem 7.1. *The cone σ^\vee for T_7 is a 9 dimensional completely split reflexive Gorenstein cone of index 3. It is the Cayley cone over three 6-dimensional lattice simplices.*

Proof. The result follows from implementing our general results from Section 4.4. I have done the computations in Maple using the package `Convex`

$$\Delta_1 : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \Delta_2 : \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & -1 & 0 & -1 \end{bmatrix} \quad \Delta_3 : \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

TABLE 1. The simplicies are the convex hull of the columns.

([Fra09]). Choose an origin in vert T . Since T_7 is chiral, \bar{M} is Gorenstein, so $\epsilon_1 + \epsilon_2 + \epsilon_3 \in M$. In fact

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = A_{p, \tau_1(p)} + A_{\tau_3(p), (\tau_2 + \tau_3)(p)} + A_{-\tau_2(p), (\tau_3 - \tau_2)(p)}.$$

As a basis for M choose

$$E_1 = A_{0, \tau_1(0)}, E_2 = A_{\tau_3(0), (\tau_2 + \tau_3)(0)}, E_3 = A_{-\tau_2(0), (\tau_3 - \tau_2)(0)} \\ E_{3+i} = m_{(i-1)\tau_1(0)}$$

for $i = 1, \dots, 6$. Thus in this basis $E_i^* = \epsilon_i^*$ for $i = 1, 2, 3$.

After expressing the columns of A in this basis, i.e. after turning M into the standard \mathbb{Z}^9 , one computes that σ^\vee is the Cayley cone over the 3 simplicies in \mathbb{R}^6 described in Table 1. Now plug this into the computer program and find that σ , the dual cone, has 24 rays, three of them are of course spanned by $\epsilon_1^*, \epsilon_2^*, \epsilon_3^*$. Moreover $m_\sigma := \epsilon_1 + \epsilon_2 + \epsilon_3 \in M$ yields the Gorenstein property on σ . This proves the theorem. \square

Corollary 7.2. *The smoothing component S is the germ of a normal Gorenstein affine toric variety.*

Proof. Since σ is Gorenstein, the point $m_\sigma := \epsilon_1 + \epsilon_2 + \epsilon_3 \in M$ generates the interior of σ^\vee , in the sense that $\text{int } \sigma \cap M = m_\sigma + \sigma \cap M$. But in this example m_σ is in the semigroup generated by the columns of A , so the semigroup ring is normal. \square

The notion of reflexive cones was introduced by Batyrev and Borisov to study mirror symmetry for complete intersection Calabi Yau manifolds in toric varieties. In our example, since σ^\vee is reflexive, the Minkowski sum $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ will be a *reflexive* 6 dimensional polytope. This polytope determines a 6 dimensional (singular) Fano variety Y . The Minkowski decomposition gives us 3 divisors E_i on Y . If we cut Y with a general section of each of the $\mathcal{O}_Y(E_i)$ the result is a singular Calabi-Yau 3-fold and one can now take a crepant resolution to arrive at a Calabi-Yau 3-manifold.

Maxmillian Kreuzer ran the polytope through the computer program PALP (see [KS04]) with the following result.

Proposition 7.3. *The Calabi Yau 3-manifold arising from the reflexive cone for T_7 has Hodge numbers $h^{1,1} = 15$ and $h^{1,2} = 12$. In particular the Euler number is 6.*

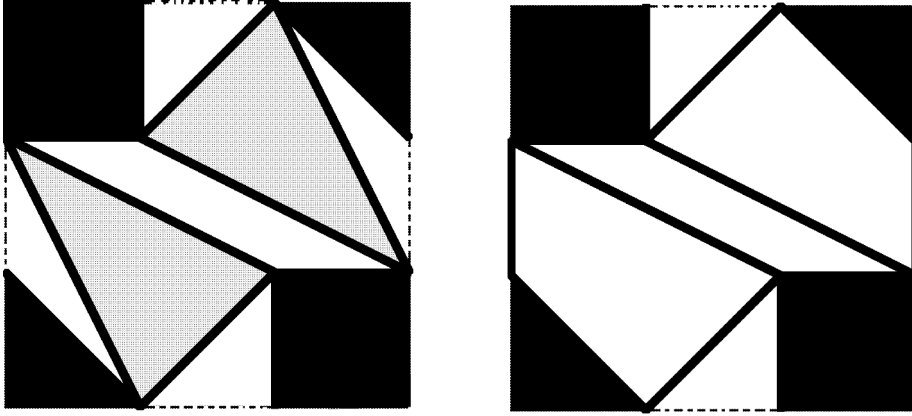


FIGURE 1. The two rigid tilings of the torus.

Note that if Z is the total space of the vector bundle $\bigoplus_{i=1}^3 \mathcal{O}_Y(E_i)$, toric geometry tells us that there is a birational toric morphism $f : (Z, Y) \rightarrow (S, 0)$, where we identify Y as the zero section.

Remark. Since the cones associated to all equivelar triangulated tori are Cayley cones, one could ask if more of them are reflexive. I have not been able to prove, but do conjecture that T_7 is the *only* polyhedral triangulation leading to a reflexive cone. There are other non-polyhedral examples.

The versal base space in this case is defined by 21 binomials in 21 variables. The numbers are small enough for us to be able to give the full component structure. This was done by delicate use of the ideal quotient command in Macaulay 2 ([GS]).

Proposition 7.4. *The versal base space $\text{Def}_{\mathbb{P}(T_7)}^a$ is reduced. It is the union of 29 irreducible components. The 28 non-smoothing components are isomorphic to the germ of the product of the affine cone over $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ and \mathbb{C}^3 or \mathbb{C}^4 .*

When one has the ideal of the component it is in this case easy to find the generic fiber. This will be an interesting scheme since it is a “generic” non-smoothable degenerate abelian surface, i.e. it cannot appear in degenerations of smooth abelian surfaces.

The 28 components come in 4 G orbits, but 3 of these (the 8 dimensional ones) have isomorphic generic fibers. In Figure 1 I have drawn two tilings of the torus, P_1 with 1 hexagon (in black), 3 quadrangles and 2 triangles and P_2 with 1 hexagon and 4 quadrangles. We can associate an embedded rational projective surface to each polygon. A hexagon corresponds to a Del Pezzo surface of degree 6 in \mathbb{P}^6 , a quadrangle corresponds to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via the Segre embedding in \mathbb{P}^3 and a triangle corresponds to \mathbb{P}^2 . Now take the union of them in \mathbb{P}^6 , intersecting as in P , to make the degenerate abelian surface X_P .

Proposition 7.5. *The generic fiber over a component of dimension 7 is X_{P_1} and over a component of dimension 8 it is X_{P_2} .*

Remark. It is probably better to think of the tilings above as periodic polygonal tilings of the plane with vertices contained in the lattice of vertices of $\{3, 6\}$. Note also that Γ is the full translation group of the tiling. All such tilings can be constructed by erasing Γ orbits of edges in $\{3, 6\}$. I believe that in general the generic fiber over a non-smoothing component may be described by erasing the edges corresponding to deformation parameters that do *not* vanish on the whole component. It is tempting to conjecture that the components of the non-smoothable fiber are the projective toric varieties associated to the $\{3, 6\}$ lattice polygons in the tiling.

Using this deformation theory we can find equations for Heisenberg invariant abelian surfaces in \mathbb{P}^6 . By Lemma 5.2 we need to find the family over the smooth subspace \mathcal{M} with three parameters $u_k = t_{p, \tau_k(p)}$.

Let us first index the vertices by their τ_1 orbit, i.e. after fixing a vertex $\{0\}$, $\{m\} = \{m\tau_1(0)\}$. Thus in cycle notation $\tau_1 = (0, 1, 2, 3, 4, 5, 6)$. Then τ_2 becomes $(0, 4, 1, 5, 2, 6, 3)$ and τ_3 is $(0, 2, 4, 6, 1, 3, 5)$. The group $\text{Aut}(T) = F_{42}$ is generated by τ_1 and the rotation $\rho = (1, 5, 4, 6, 2, 3)$. Note that ρ acts on \mathcal{M} as the permutation (u_1, u_3, u_2) .

The 21 cubic monomials generating I_X are one F_{42} orbit, but it is convenient to partition them in 3 τ_1 orbits since they have the nice form

$$x_{-\tau_k(p)}x_px_{\tau_k(p)} \quad k = 1, 2, 3 \quad p \in \text{vert } T.$$

Note again that ρ permutes these 3 orbits.

The first order deformations are easily found and the first order family is defined by the τ_1 orbits of

$$x_0x_1x_6 + u_2(x_1^2x_5 + x_2x_6^2), x_0x_3x_4 + u_3(x_4^2x_6 + x_1x_3^2), x_0x_2x_5 + u_1(x_2^2x_3 + x_4x_5^2).$$

Instead of lifting equations and relations to continue apply the symmetry.

The ideal must be Heisenberg invariant. In our case the action of G^* is generated by $x_m \mapsto \zeta_7^m x_m$. Thus if $x_{m_1}x_{m_2}x_{m_3}$ is a term in the perturbation of $x_{-\tau_k(m)}x_mx_{\tau_k(m)}$ we must have

$$m_1 + m_2 + m_3 \equiv -\tau_k(m) + m + \tau_k(m) \pmod{7}.$$

Moreover $\rho^3 = (1, 6)(2, 5)(3, 4)$ fixes each $x_{-\tau_k(m)}x_mx_{\tau_k(m)}$, so terms in ρ^3 orbits in the perturbation must have the same coefficient. Finally we may exclude terms that are in I_X . The upshot is that the family is defined by

the orbits of

$$\begin{aligned}
 (7.1) \quad & x_0x_1x_6 + u_2(x_1^2x_5 + x_2x_6^2) + \psi_1(x_1x_2x_4 + x_3x_5x_6) \\
 & + \varphi_1x_0^3 + \xi_1(x_1x_3^2 + x_4^2x_6) + v_1(x_2^2x_3 + x_4x_5^2) \\
 & x_0x_3x_4 + u_3(x_4^2x_6 + x_1x_3^2) + \psi_2(x_1x_2x_4 + x_3x_5x_6) \\
 & + \varphi_2x_0^3 + \xi_2(x_4x_5^2 + x_2^2x_3) + v_2(x_1^2x_5 + x_2x_6^2) \\
 & x_0x_2x_5 + u_1(x_2^2x_3 + x_4x_5^2) + \psi_3(x_1x_2x_4 + x_3x_5x_6) \\
 & + \varphi_3x_0^3 + \xi_3(x_2x_6^2 + x_1^2x_5) + v_3(x_4^2x_6 + x_1x_3^2)
 \end{aligned}$$

where the $\xi_i, v_i, \varphi_i, \psi_i$ are power series in u_1, u_2, u_3 . Thus the task becomes to find similar expressions for lifted relations and then solving functional equations to make the family flat.

I describe here the answer only for the one parameter deformation $s = u_1 = u_2 = u_3$. In this case also $\psi_1 = \psi_2 = \psi_3$ etc. so denote the common function by ψ . Let $f(s)$ be a power series solution for the equation

$$s^6 f(s)^4 - s^4(s+1)f(s)^3 - (s+1)^2(s-1)f(s) + (s+1)^2 = 0.$$

Proposition 7.6. *The family defined by the orbits of equations (7.1) form a flat one parameter smoothing if $s = u_1 = u_2 = u_3$ and*

$$\varphi = -\psi = \frac{s^2(s^4 f(s)^3 - s - 1)}{1 + s}, \quad \xi = s^2 f(s), \quad v = \frac{s^4 f(s)^2}{1 + s}.$$

Because of the symmetry only 2 relations need to be lifted. I computed the liftings using Maple.

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